

## Research Article

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# Entropy generation and heat transfer in nonlinear Buoyancy-driven Darcy–Forchheimer hybrid nanofluids with activation energy

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**Abstract:** This study investigates the influence of a magnetic field, activation energy, and heat source on the heat and mass transfer within a cross fluid embedded with mono-, di-, and tri-nanoparticles, considering thermal radiation and Darcy–Forchheimer effects. Utilizing the Cattaneo–Christov theory, non-Fourier heat transfer is modeled for a vertical moving surface. A mathematical model is developed and subsequently converted into a dimensionless form through an appropriate similarity transformation, resulting in a system of first-order ordinary differential equations. The numerical approach to solve the system is BVP4C solver in MATLAB, a tool specifically designed for boundary value problems. Graphical representations have been analyzed for velocity profiles, temperature profiles, and concentration distributions for different values of physical parameters. It is observed that the velocity profiles exhibit an upward trend with an increase in the parameters associated with nonlinear thermal convection and nonlinear concentration convection. Additionally, the analysis of surface shear stress, heat transfer coefficients, and mass transfer coefficients revealed that an

increase in the porosity parameter and Forchheimer number results in decreased shear stress. Entropy generation is also investigated to quantify irreversibilities in the system. The analysis showed that increasing the Brinkman number, diffusion parameter, and temperature and concentration difference parameters leads to higher entropy generation, indicating greater irreversibility in the system. A comparative analysis demonstrates that tri-nanoparticles substantially improve flow velocity, thermal conductivity, and solute diffusion compared to di- and mono-nanoparticles, with tri-nanofluids exhibiting the most optimal overall performance.

**Keywords:** Darcy–Forchheimer, porous medium, cross fluid, activation energy, hybrid nanofluid, heat transfer

## Nomenclature

|            |                                                             |
|------------|-------------------------------------------------------------|
| $A$        | ratio parameter                                             |
| $B_0$      | magnetic field                                              |
| $Br$       | Brinkmann number                                            |
| $G_1$      | stretching rate                                             |
| $G_2$      | rate of free stream velocity                                |
| $C_f$      | drag coefficient                                            |
| $C_p$      | specific heat at constant pressure ( $J\ kg^{-1}\ K^{-1}$ ) |
| $C_\infty$ | ambient concentration ( $mol/m^3$ )                         |
| $C_w^*$    | concentration at the wall ( $mol/m^3$ )                     |
| $D$        | diffusion coefficient                                       |
| $Ec$       | Eckert number                                               |
| $g$        | gravity ( $m/s^2$ )                                         |
| $Gr$       | Grashof number                                              |
| $Gc$       | Solutal Grashof number                                      |
| $Ha$       | Hartmann number                                             |
| $k$        | thermal conductivity ( $W/(m\ K)$ )                         |
| $m$        | rheological power index                                     |

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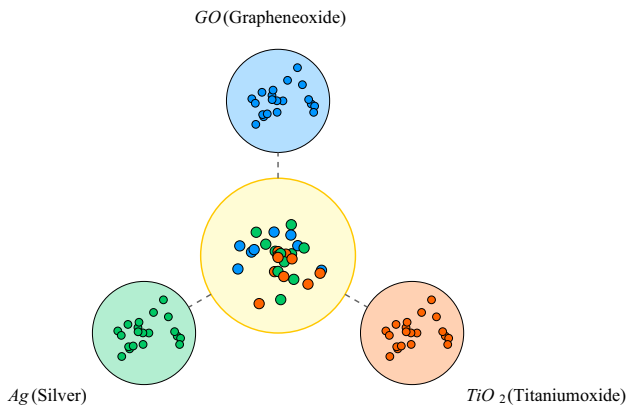
|               |                                               |
|---------------|-----------------------------------------------|
| $m_w$         | mass flux (kg/s)                              |
| Nu            | Nusselt number                                |
| Pr            | Prandtl number                                |
| $q_r$         | heat flux (W/m <sup>2</sup> )                 |
| Re            | Reynolds number                               |
| Sh            | local Sherwood number                         |
| Sc            | Schmidt number                                |
| $T_w^*$       | wall temperature (K)                          |
| $T_\infty^*$  | fluid ambient temperature (K)                 |
| $T^*$         | fluid temperature                             |
| $u, v$        | velocity along $x$ and $y$ directions         |
| $U_w$         | free stream velocity (m/s)                    |
| $U_s$         | stretching velocity of sheet (m/s)            |
| $x, y$        | Cartesian coordinates                         |
| We            | Weissenberg number                            |
| $\beta_1^*$   | linear thermal expansion coefficient          |
| $\gamma_1^*$  | linear concentration expansion coefficient    |
| $\beta_2^*$   | nonlinear thermal expansion coefficient       |
| $\gamma_2^*$  | nonlinear concentration expansion coefficient |
| Greek letters |                                               |
| $\delta$      | solutal relaxation parameter                  |
| $\delta_1$    | thermal relaxation parameter                  |
| $\eta$        | similarity variable                           |
| $\Gamma$      | time constant                                 |
| $\lambda_1$   | thermal relaxation time                       |
| $\mu$         | dynamic viscosity                             |
| $\nu$         | kinematic viscosity                           |
| $\phi$        | dimensionless concentration                   |
| $\rho$        | fluid's density (kg/m <sup>3</sup> )          |
| $\sigma$      | electrical conductivity (S/m)                 |
| $\tau_w$      | wall shear stress (Pa)                        |
| $\theta$      | dimensionless temperature                     |
| Subscripts    |                                               |
| $f$           | dimensionless velocity                        |
| $dnf$         | di-nanofluid                                  |
| $nf$          | nanofluid                                     |
| $tnf$         | tri-nanofluid                                 |

## 1 Introduction

The transfer of heat and mass through fluids is a significant factor in many scientific and engineering fields. This process affects daily activities such as conduction, convection,

and evaporation, and is essential in industries including distillation, extraction, automotive engineering, aerospace, and power generation. In both heating and cooling systems, the efficiency of heat transfer largely depends on thermal diffusivity, a key parameter determining how quickly heat is conducted. Increasing thermal diffusivity enhances heat and mass transfer, as higher diffusivity leads to a faster rate of heat conduction. Various strategies have been explored by researchers to improve thermal conductivity, with a particular emphasis on using nanoparticles. Nanoparticles, which can consist of metals, oxides, carbides, or carbon nanotubes, have been proven to enhance the thermal properties of fluid. Nanofluids, which are created by dispersing nanoparticles into base fluids, have demonstrated significant improvements in the thermal conductivity of these fluids. This study focuses on investigating how the thermal conductivity of a cross fluid changes with the dispersion of tri-nanoparticles specifically graphene oxide (GO), silver (Ag), and titanium dioxide (TiO<sub>2</sub>), (Figure 1) within a porous medium. Additionally, the study will compare these results with the variations observed when di-nano and mono-nano particles are used in the same system, providing insight into the impact of different nanoparticle combinations on thermal behavior. Sharma *et al.* [1] performed a computational analysis on hybrid nanoparticle flow (Au-Al<sub>2</sub>O<sub>3</sub> in blood) in a curved artery under stenosis and aneurysm conditions, analyzing heat and mass transport. Tiwari and Das [2] explored the improvement of thermal performance in a thermally varied square cavity with dual lid-driven boundaries using nanofluids. Nawaz *et al.* [3] examined the effects of variable thermal diffusivity concerning the circulation of micropolar nanofluids generated by a stretchable disk, considering Arrhenius activation energy. Annapureddy *et al.* [4] studied heat and mass transport in the unsteady magnetohydrodynamic (MHD) flow of a Maxwell fluid over a vertically extending porous sheet, including activation energy and thermal diffusion effects. Li *et al.* [5] performed a statistical and morphological analysis of cross-rheological materials transporting alumina, silica, titanium, and ethylene glycol using the Galerkin method.

Several researchers have utilized mono and di-nano particles to improve the properties of base fluids. Sadiq [6] studied the enhancement of non-Fourier heat transfer in power-law fluids using both mono and hybrid nanoparticles. Kaneez *et al.* [7] focused on improving heat transport in yield-stress dusty fluids by incorporating mono and hybrid nanoparticles. Yaseen *et al.* [8] explored thermal transfer in the behavior of mono-nanofluids and hybrid nanofluids in flow conditions between parallel plates, considering thermal radiation and heat generation/absorption within a Darcy porous medium. Kanthimathi *et al.* [9]



**Figure 1:** Hybrid nanoparticle mixtures.

analyzed the thermophysical properties and heat transfer behavior of mono and hybrid nanofluids using different base fluids. Gangadhar *et al.* [10] analyzed thermal radiation effects on Joule heating for wall jet blood flow in the presence of hybrid nanofluid over moving surface. Shaheen *et al.* [11] investigated the flow of  $\text{Fe}_3\text{O}_4\text{--ZrO}_2$  and  $\text{Fe}_3\text{O}_4/\text{water}$  under the influence of heat production, nonlinear thermal radiation and nanoparticle volume fractions. Gangadhar *et al.* [12] investigated the viscoelastic Oldroyd-B nanofluid flow through a vertical stretching sheet with swimming gyrotactic microorganisms. Elattar *et al.* [13] examined the nonlinear heat source/sink influence on the 3D flow of water-based silver nanoparticles incorporated in an Eyring-Powell fluid with concentration pollutants. Shaheen *et al.* [14] studied a statistical optimization of copper and alumina nanoparticles and entropy generation through a stretching and shrinking inclined surface. Naqvi *et al.* [15] established a mathematical model for entropy generation in MHD nanofluid over both stretched and shrinking surfaces. Shaheen *et al.* [16] studied the influence of a magnetic field and thermal radiation for scrutinizing the MHD hybrid nanofluid in two parallel plates using suction and injection walls. Khan *et al.* [17] investigated the wall jet flow and heat transfer through the Eyring-Powell nanofluid and significant consequence of erratic heat source/sink and buoyancy. Ali *et al.* [18] inspected the MHD flow of Powell-Eyring fluid induced by the nanofluid and bioconvection flow through a spinning disk. Khan *et al.* [19] explored the melting process of MHD hybrid AA7072 and AA7075 alloy nanoparticles from a movable cylinder. Li *et al.* [20] presented the thermo-diffusion applications of nanofluid subject to variable thermal sources. Panda *et al.* [21] studied the influence of convective heating on the fluid flow of a three-dimensional ferromagnetic Casson hybrid nanofluid over a radiative Riga sensor device. For additional insights, refer to the works of Rajesh *et al.* [22], Bhattad *et al.* [23], and Xiao *et al.* [24]. In recent years, the dispersion of tri-nanoparticles has emerged

as a significant method for enhancing heat transfer. This research investigates the dispersion of tri-nanoparticles in a cross fluid, focusing on the role of the Darcy–Forchheimer model and activation energy. The Darcy–Forchheimer model is widely used to describe fluid flow through porous media and is applicable in areas such as medicine, soil contamination, and chemical engineering. This model is particularly useful in predicting high-velocity flow in porous structures. Abbas *et al.* [25] examined the impact of the Darcy–Forchheimer model on MHD dissipative analysis of flow behavior and heat transfer in a third-grade fluid across a porous substrate, including Joule heating effects. Xiong *et al.* [26] studied the dynamics of multiple solutions in cross-nanofluid flow with a Darcy–Forchheimer saturated medium, driven by a vertical thin needle. Additionally, Sharma *et al.* [27] investigated the effects of Joule heating and nonuniform heat sources/sinks in unsteady MHD mixed convection flow along a vertically deforming surface with a Darcy–Forchheimer porous medium.

Activation energy, defined as the minimum energy required to initiate a chemical reaction, plays a crucial role in enhancing mass transfer. It also contributes to improvements in thermal conductivity, influencing temperature profiles and thermal boundary layer thickness. Sharma *et al.* [28] explored the influence of Arrhenius activation energy on MHD flow of gyrotactic microorganisms in a porous medium, considering thermophoresis and Brownian motion. Gangadhar *et al.* [29] analyzed two-dimensional unsteady squeezing Casson fluid through porous channel with activation. Jabeen *et al.* [30] analyzed the importance of activation energy in the stratified flow of tangent hyperbolic fluids, while Hassan *et al.* [31] examined its effects on Maxwell nanofluid flow through a porous medium. The cross fluid in this study is a non-Newtonian fluid, characterized by variable viscosity dependent on applied stress, which significantly influences its flow behavior. Non-Newtonian fluids exhibit nonlinear flow behavior, allowing better flow control and reducing viscosity during flow, thereby enhancing mixing and improving flow through complex geometries. Many studies have focused on the properties and behavior of non-Newtonian fluids. This research further explores these dynamics by analyzing tri-nanoparticle dispersion in a base fluid composed of ethylene glycol and water. For more comprehensive discussions, consult Borjigin *et al.* [32], Hussain *et al.* [33], Hauswirth *et al.* [34], and Reddy *et al.* [35]. Buoyancy, which is the force generated by a fluid that opposes the weight of an object, also plays a significant role in fluid flow. It helps explain whether an object floats or sinks and is fundamental in understanding phenomena such as the buoyancy of boats or the ease of lifting objects underwater.

Kumar and Sood [36] studied the interaction between a magnetic field and nonlinear convection in stagnation point flow over a shrinking sheet. Manzur *et al.* [37] analyzed the effects of buoyancy-assisted and opposing flows on mixed convection heat transfer in a cross fluid with thermal radiation. Ayub *et al.* [38] explored heat transport on a heated surface under thermal radiation and MHD cross flow, with a focus on nonuniform heat sinks/sources and buoyancy-driven flows. Hafeez *et al.* [39] studied the effects of buoyancy on chemically reactive cross-nanofluid flow patterns on a surface that is contracting. Irreversibility is inherent in all thermofluid processes, reducing system efficiency. Entropy generation is used to quantify this irreversibility and is vital in analyzing energy conversion processes. Lopez *et al.* [40] investigated entropy generation in MHD nanofluid flow in a porous vertical microchannel, accounting for nonlinear thermal radiation, slip flow, and convective-radiative boundary conditions. Khan *et al.* [41] studied entropy optimization in Williamson nanofluid flow, considering chemical reactions and Joule heating effects. Narla *et al.* [42] explored entropy generation in electroosmotic pumping of nanofluids through curved channels, considering Joule dissipation. Gangadhar and Chamkha [43] investigated the entropy generation for the magnetized coupled stress fluid passed through a permeable stretching cylinder. For additional insights into entropy, refer to the studies by Sharma *et al.* [44,45], Abbas *et al.* [46], and Alzahrani and Khan [47]. This study focuses on the analysis of heat and mass transport in a cross fluid across a stretched surface, incorporating boundary layer flow. Cross fluids, particularly those with shear-thinning properties, are commonly used in industries such as polymers, paints, inks, food products, and biological fluids. Understanding their flow behavior is essential for applications such as pumping, spraying, and mixing. Hosseinzadeh *et al.* [48] studied about cross fluid dynamics involving motile gyrotactic microorganisms and nanoparticles near a three-dimensional cylinder. Nazeer *et al.* [49] examined numerical and perturbation analyses of the cross flow behavior of an Eyring-Powell fluid. Exploration of nonlinear dynamics in a fluid-conveying pipe subjected to the combined effects of cross-flow and top-end excitations is observed by He *et al.* [50], analysis of turbulent cross-flow in a staggered tube array through experimental and numerical approaches was implemented by Paul *et al.* [51]. Kotha *et al.* [52] studied the two-dimensional MHD flow and heat and mass transfer phenomena of water-based nanofluid containing gyrotactic microorganisms over a vertical plate. Boundary layer flow occurs due to which fluid velocity changes from zero to the free stream-velocity. Boundary layer flow is a fundamental idea

in fluid dynamics that has an impact on many different fields, including environmental engineering and aircraft design. Efficiency, performance, and safety can all be significantly increased in a variety of engineering systems by comprehending and managing boundary layer behavior. Engineers can create more efficient solutions to challenging fluid flow problems by researching the properties and applications of boundary layers. Tan *et al.* [53] studied stress-sensitive permeability in porous media with fluid–solid coupling through a fractal geometry-based model. Recent advancements by Xiong *et al.* [54] and Keller and Cebeci [55] have significantly shaped the understanding of boundary layer.

A comprehensive review of existing literature shows that no mathematical investigations have yet been conducted on non-Fourier thermal and material transport within cross fluids, especially through a two-dimensional, vertically contracting surface interacting with porous medium. This study also incorporates the effects of activation energy with chemical reactions. Additionally, it includes Darcy–Forchheimer model to characterize fluid flow through porous media, accounting for buoyancy, heat sources, and thermal radiation, thereby providing a comprehensive framework for complex fluid behaviors. The governing partial differential equations are reduced to a set of ordinary differential equations through a similarity transformation. The resulting dimensionless coupled nonlinear boundary value problem is then solved using the BVP4C method in MATLAB. Detailed tabular and graphical results are presented for parameters such as thermal radiation, Prandtl number, heat source strength, Schmidt number, chemical reaction rate, mixed convection, nonlinear temperature convection, nonlinear concentration convection, flow speed, thermal profile, and concentration distributions. The study also investigates skin drag, and both the Nusselt and Sherwood numbers. The comparative analysis highlights the role of nanoparticle composition in influencing fluid properties, while also emphasizing the potential of tri-nanoparticle systems to enhance thermal management. The structure of this study is divided into six sections: “Introduction,” “Problem Model,” “Numerical Methodology,” “Comparison,” “Entropy Generation,” and “Conclusion.” Each section methodically addresses critical aspects of the study, ensuring a well-organized analysis of the research topic.

The Forchheimer number quantifies the importance of inertial forces in the flow of a fluid through a porous medium. It represents the nonlinear drag effect, indicating how the resistance to fluid flow increases with velocity. Activation energy defines the threshold energy required for a thermally activated process to occur. It controls the rate at which a process, such as heat transfer or mass transfer, with higher activation energy leads to slower

reactions or transitions at lower temperatures. This research introduces a novel approach to heat transfer modeling by utilizing the Cattaneo–Christov theory for non-Fourier heat transfer, offering a more precise representation of thermal behavior than conventional Fourier-based models. No attempts have been made to analyze the entropy generation of the MHD cross fluid flow with hybrid nanoparticles in heat and mass transfer processes via vertical surfaces stretching under the influence of solar radiations. Compared to previous studies, this research highlights that tri-nanoparticles significantly enhance flow velocity, thermal conductivity, and solute diffusion, outperforming mono- and di-nanoparticle systems. The inclusion of entropy generation analysis allows us to quantify system inefficiencies, offering essential insights into the optimization of thermal systems for energy efficiency. The work uniquely addresses the intersection of non-Fourier heat transfer, tri-nanoparticle fluid systems, and activation energy effects, creating a distinctive contribution to the field of heat and mass transfer in nanofluids. The findings of this study have important applications in various engineering and industrial fields. The improved heat transfer in tri-nanoparticle-enhanced fluids makes them suitable for use in advanced thermal management systems, renewable energy technologies, and industrial heat exchangers. These fluids can help improve energy storage systems, increase the efficiency of chemical and process operations, and support the development of sustainable and eco-friendly technologies. Additionally, the study provides valuable insights into fluid behavior under Darcy–Forchheimer effects and activation energy, which can help design more efficient cooling systems for electronics and renewable energy devices, leading to better performance and reduced energy waste.

1. What are the key benefits of tri-nanoparticles over mono- and di-nanoparticles in enhancing thermal and mass transfer in hybrid nanofluids?
2. How do activation energy and Darcy–Forchheimer effects improve the understanding of heat and mass transfer in nanofluids for practical applications?
3. How can entropy generation insights guide the development of more efficient thermal management systems in industrial settings?
4. What insights does the Bejan number provide into the irreversibility of heat transfer in hybrid nanofluids, and how can it aid in enhancing energy efficiency?

## 2 Model formulation and analysis

This work examines the steady-state behavior of incompressible cross-fluid flow (non-Newtonian) over a

vertically stretched surface, considering the combined effects of Joule heating, relaxation time, and thermal radiation on thermal and mass transport due to the dispersion of tri-nanoparticles. The flow is assumed to be laminar and two-dimensional, with the base fluid being a mixture of water and ethylene glycol. The nanoparticles are evenly distributed throughout the base fluid, without any aggregation or settling, fluid and nanoparticle's physical properties are considered invariant throughout the study.

For simplification, the analysis is based on the following key assumptions:

- 1) The study employed the Cattaneo–Christov framework to explore non-Fourier heat and mass transfer processes in a cross-fluid system.
- 2) The fluid flow with the velocity of wall, i.e.,  $U_s$  also depends on the properties of nanoparticles that are dispersed.
- 3) Go, Ag,  $\text{TiO}_2$  nanoparticles are considered for visualizing enhancement of heat and mass transfer in cross fluid. Darcy–Forchheimer with porous medium, activation energy, and buoyancy force are also taken into account for the analysis of heat and mass transfer.
- 4) The symbols  $T_s^*$  and  $C_s^*$  are used to denote the surface temperature and surface concentration. Conversely, the ambient (external) temperature and concentration are symbolized by  $T_\infty^*$  and  $C_\infty^*$ .
- 5) External temperature and concentration are constant whereas surface temperature and surface concentration are variables
- 6) Boundary layers are generated that play a significant role in heat and mass transfer (Figure 2).

The mathematical modeling of the governing equations is derived from the conservation principles of mass,

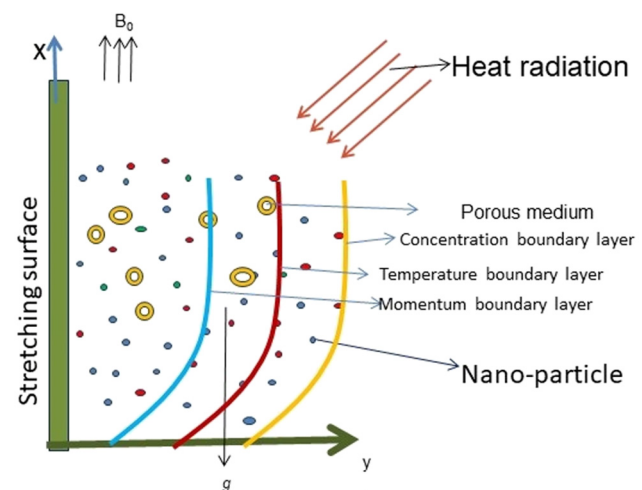


Figure 2: Physical geometry of the current model.

momentum, energy, and concentration, incorporating the effects of nanoparticle concentration. In this study, the nanoparticles under consideration are GO, Ag, and TiO<sub>2</sub>, which influence the thermal and mass transport characteristics of the system.

Under all these considerations the Governing model modification equations [56–58] of momentum, thermal state, and concentration, along with boundary conditions are as follows:

#### Continuity equation

$$u_x + v_y = 0. \quad (1)$$

#### Momentum equation

$$\begin{aligned} uu_x + vv_y = & v_{tnf} \left( \frac{u_y}{1 + (\Gamma u_y)^m} \right)_y + U_e (U_e)_x \\ & + \frac{\sigma_{tnf} B_0^2}{\rho_{tnf}} (U_e - u) - \frac{\mu_{tnf}}{\rho_{tnf} k} u \\ & - \frac{c_b u^2}{\sqrt{k}} + g [\beta_1^* (T^* - T_\infty^*) + \beta_2^* (T - T_\infty^*)^2] \\ & + g [\gamma_1^* (C^* - C_\infty^*) + \gamma_2^* (C^* - C_\infty^*)^2]. \end{aligned} \quad (2)$$

#### Energy equation

$$\begin{aligned} uT_x^* + vT_y^* + \lambda_1 [uu_x T_x^* + vv_y T_y^* + uv_x T_y^* + uv_y T_x^* \\ + 2uvT_{xy}^* + u^2 T_{xx}^* + v^2 T_{yy}^*] \\ = \frac{k_{tnf}}{(\rho c_p)_{tnf}} T_{yy}^* + \frac{Q_0 (T^* - T_\infty^*)}{(\rho c_p)_{tnf}} - \frac{1}{(\rho c_p)_{tnf}} \left( \frac{\partial q_r}{\partial y} \right). \end{aligned} \quad (3)$$

#### Concentration equation

$$\begin{aligned} uC_x^* + vC_y^* + \lambda_2 [uu_x C_x^* + vv_y C_y^* + uv_x C_y^* + uv_y C_x^* \\ + 2uvC_{xy}^* + u^2 C_{xx}^* + v^2 C_{yy}^*] \\ = D_{tnf} C_{yy}^* - k_r^2 (C^* - C_\infty^*) \left( \frac{T^*}{T_\infty^*} \right)^2 \exp \left( -\frac{E_a}{k_e T^*} \right). \end{aligned} \quad (4)$$

#### Boundary condition

$$\begin{aligned} u = U_\omega = c_1 x, \quad v = 0, \quad C^* = C_w^*, \quad T^* = T_\omega^* \quad \text{at } y = 0 \\ u \rightarrow U_e = c_2 x, \quad v = 0, \quad T^* \rightarrow T_\infty^*, \quad C^* \rightarrow C_\infty^* \quad \text{as } y \rightarrow \infty, \end{aligned}$$

where  $x$  and  $y$  are spatial coordinates.  $u$  and  $v$  are velocity components along  $x$  and  $y$  directions.  $T$  indicates the fluid's temperature at a specific location, providing insight into the heat distribution and thermal behavior within the system. Similarly,  $C$  denotes the concentration of solutes or nanoparticles at a particular point, describing how these components are spatially distributed throughout the fluid. Equation (1) is the continuity equation for two-dimensional steady-state hybrid nanofluid flow which preserves mass conservation

principle, equation (2) is the momentum equation representing velocity distribution in the fluid flow with convection, diffusion, and magnetic field terms, equation (3) is the energy equation representing the thermal distribution in the hybrid nanofluid flow due to the convection, and radiation heat transfer, equation (4) is the concentration equation governed by convection and diffusion, and equation (5) represents  $q_r$ , which is the nonuniform radiative heat flux distribution. These are the boundary conditions for stretching surface defining the velocity and temperature behavior. The velocity is set as  $u = U_\omega(x)$ , reflecting the stretching surface's motion, while the temperature is set as  $T^* = T_\omega^*(x)$ , capturing the thermal distribution. These conditions are essential to accurately model the physical interactions at the stretching surface for  $y = 0$ , ensuring meaningful analysis of fluid flow and heat transfer. At  $y \rightarrow \infty$ , the conditions are far from the surface, where the fluid velocity approaches the free-stream velocity, and temperature and concentration stabilize to ambient levels.

For simplification, boundary layer equations are made dimensionless via suitable transformation.

Here  $u_x$  represents partial derivative of  $u$  with respect to (w.r.t)  $x$  and  $v_y$  represents velocity components of  $v$  w.r.t  $y$ .  $\nu$  represents the kinetic viscosity,  $m$  represents the rheological power index,  $U_e$  denotes the external velocity,  $g$  denotes the gravitational acceleration,  $Q_0$  stands for the heat generation coefficient,  $q_r$  denotes the radioactive heat flux,  $k_r^2$  denotes the chemical reaction parameter,  $c_b$  the drag coefficient,  $\beta_1^*$  and  $\beta_2^*$  denote the thermal expansion coefficients,  $\gamma_1^*$  and  $\gamma_2^*$  denote the concentration expansion coefficients,  $\lambda$  denotes the porosity parameter,  $E_a$  denotes the activation energy, and  $\left( \frac{T^*}{T_\infty^*} \right)^2 \exp \left( -\frac{E_a}{k_e T^*} \right)$  represents the Arrhenius function where  $a$  is the positive constant. Radioactive heat flux is defined as using Rosseland approximation [59]

$$q_r = \frac{-4\sigma^*}{3k^*} \frac{\partial T^{*4}}{\partial y}, \quad (5)$$

where  $\sigma^*$  represents Stefan Boltzmann constant and  $k^*$  represents the Rosseland mean absorption coefficient. To solve the term  $T^{*4}$ , Taylor series expansion about an ambient temperature is defined as

$$T^{*4} = T_\infty^{*4} + 4T_\infty^{*3}(T^* - T_\infty^*) + 6T_\infty^{*2}(T^* - T_\infty^*)^2 + \dots$$

Neglecting all higher order terms and taking only linear term in  $(T^* - T_\infty^*)$ , approximate value of  $T^{*4}$  becomes

$$T^{*4} = 4T_\infty^{*3}T^* + T_\infty^{*4}. \quad (6)$$

By using equation (6) in (5), value of  $q_r$  is obtained and then  $\frac{\partial q_r}{\partial y}$  as follows:

$$\frac{\partial q_r}{\partial y} = \frac{-16\sigma^* T_\infty^{*4}}{3k^*} \frac{\partial^2 T^*}{\partial y^2}. \quad (7)$$

### 3 Nonsimilarity transformation

Nonsimilarity solutions hold significant importance in fluid mechanics due to their accuracy and ability to simplify independent variables. These solutions are often derived as asymptotic solutions, providing insightful interpretations of intricate fluid dynamics phenomena. They effectively capture the prevailing dynamic, thermal, and physical conditions while illustrating their impact. The nonsimilar variables [60] utilized for formulating the dimensionless governing equations are

$$\begin{cases} u = c_1 x \frac{\partial f(\xi, \eta)}{\partial \eta}, \\ v = -\sqrt{\frac{c_1}{\nu_f}} \left( f(\xi, \eta) + \xi \frac{\partial f(\xi, \eta)}{\partial \xi} \right), \\ \theta(\xi, \eta) = \frac{T^* - T_\infty^*}{T_w^* - T_\infty^*}, \\ \phi(\xi, \eta) = \frac{C^* - C_\infty^*}{C_w^* - C_\infty^*}, \\ \xi = \frac{x}{l}, \\ \eta = \sqrt{\frac{c_1}{\nu_f}} y, \end{cases} \quad (8)$$

where  $(\phi)$ ,  $(\theta)$ ,  $(\xi)$ , and  $(\eta)$  are dimensionless quantities. By applying the given nonsimilar variables, we derive the dimensionless forms of the governing equations.

#### Dimensionless equations obtained are

$$\begin{aligned} & \frac{\nu_{mf}}{\nu_f} \left( 1 + (1-m) \left( \text{We} \frac{\partial^2 f}{\partial \eta^2} \right)^m \right) \frac{\partial^3 f}{\partial \eta^3} \\ & + \left[ f \frac{\partial^2 f}{\partial \eta^2} + \xi \left( \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \xi \partial \eta} \right) - \left( \frac{\partial f}{\partial \eta} \right)^2 \right. \\ & - \frac{\sigma_{mf}}{\sigma_f} \frac{\rho_f}{\rho_{mf}} \left[ A^2 + \left( A - Ha^2 \frac{\partial f}{\partial \eta} \right) \right] \\ & - K \frac{\partial f}{\partial \eta} - F \xi \left( \frac{\partial f}{\partial \eta} \right)^2 + L^* \theta (1 + G\theta) + N\phi (1 + M\phi) \Big] \\ & \left( 1 + \left( \text{We} \frac{\partial^2 f}{\partial \eta^2} \right)^m \right)^2 = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} & \left( \frac{k_{mf}}{k_f} + \frac{4}{3} \text{Rd} \right) \frac{\partial^2 \theta}{\partial \eta^2} - \delta \text{Pr} \left[ f^2 + 2f\xi \frac{\partial f}{\partial \xi} + \xi^2 \left( \frac{\partial f}{\partial \xi} \right)^2 \right] \frac{\partial^2 \theta}{\partial \eta^2} \\ & + \delta \text{Pr} \left[ 2\xi \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \xi} + \xi^2 \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \xi^2} - f \frac{\partial f}{\partial \eta} - \xi f \frac{\partial^2 f}{\partial \eta \partial \xi} \right. \\ & - \xi \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \xi} - \xi^2 \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \xi \partial \eta} \Big] \frac{\partial \theta}{\partial \eta} - \text{Pr} \left[ \xi \left( \frac{\partial f}{\partial \eta} \frac{\partial \theta}{\partial \xi} \right. \right. \\ & - \left. \left. \frac{\partial f}{\partial \eta} \frac{\partial \theta}{\partial \xi} \right) - f \frac{\partial \theta}{\partial \eta} \right] + \delta \text{Pr} \left[ 2\xi f \frac{\partial f}{\partial \eta} \frac{\partial^2 \theta}{\partial \xi \partial \eta} \right. \\ & + 2\xi^2 \frac{\partial f}{\partial \xi} \frac{\partial f}{\partial \eta} \frac{\partial^2 \theta}{\partial \xi \partial \eta} + \xi^2 \frac{\partial f}{\partial \xi} \frac{\partial \theta}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} + f \xi \frac{\partial \theta}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} \\ & - \left. \xi^2 \frac{\partial f}{\partial \eta} \frac{\partial \theta}{\partial \xi} \frac{\partial^2 f}{\partial \xi \partial \eta} - \xi \left( \frac{\partial f}{\partial \eta} \right)^2 \frac{\partial \theta}{\partial \xi} \right] + Q \text{Pr} \theta = 0 \end{aligned} \quad (10)$$

$$\begin{aligned} & \frac{D_{mf}}{D_f} \frac{\partial^2 \phi}{\partial \eta^2} - \delta_1 \text{Sc} \left[ f^2 + 2f\xi \frac{\partial f}{\partial \xi} + \xi^2 \left( \frac{\partial f}{\partial \xi} \right)^2 \right] \frac{\partial^2 \phi}{\partial \eta^2} \\ & + \delta_1 \text{Sc} \left[ 2\xi \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \xi} + \xi^2 \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \xi^2} - f \frac{\partial f}{\partial \eta} \right. \\ & - \left. \xi f \frac{\partial^2 f}{\partial \eta \partial \xi} - \xi \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \xi} - \xi^2 \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \xi \partial \eta} \right] \frac{\partial \phi}{\partial \eta} \\ & - \text{Sc} \left[ \xi \left( \frac{\partial f}{\partial \eta} \frac{\partial \phi}{\partial \xi} - \frac{\partial f}{\partial \eta} \frac{\partial \phi}{\partial \xi} \right) - f \frac{\partial \phi}{\partial \eta} \right] \\ & + \delta_1 \text{Sc} \left[ 2\xi f \frac{\partial f}{\partial \eta} \frac{\partial^2 \phi}{\partial \xi \partial \eta} + 2\xi^2 \frac{\partial f}{\partial \xi} \frac{\partial f}{\partial \eta} \frac{\partial^2 \phi}{\partial \xi \partial \eta} \right. \\ & + \xi^2 \frac{\partial f}{\partial \xi} \frac{\partial \phi}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} + f \xi \frac{\partial \phi}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} - \xi^2 \frac{\partial f}{\partial \eta} \frac{\partial \phi}{\partial \xi} \frac{\partial^2 f}{\partial \xi \partial \eta} \\ & - \left. \xi \left( \frac{\partial f}{\partial \eta} \right)^2 \frac{\partial \phi}{\partial \xi} \right] - K_E \text{Sc} (1 + \theta \alpha_1)^m \phi \exp \left( \frac{-E}{1 + \theta \alpha_1} \right) \\ & = 0, \end{aligned} \quad (11)$$

along with boundary conditions

$$\begin{aligned} & \text{At } \eta = 0 : \quad f(\xi, \eta) = -\xi \frac{\partial f}{\partial \xi}, \quad \frac{\partial f(\xi, \eta)}{\partial \eta} = 1, \\ & \theta(\xi, \eta) = 1, \quad \phi(\xi, \eta) = 1 \\ & \text{When } \eta \rightarrow \infty : \quad \frac{\partial f(\xi, \eta)}{\partial \eta} \rightarrow A, \\ & \theta(\xi, \eta) = \phi(\xi, \eta) \rightarrow 0. \end{aligned} \quad (12)$$

where the prime symbol ( $'$ ) represents the derivative with respect to  $\eta$  and the nondimensional parameter we obtained are

Weissenberg number, magnetic number, Prandtl number, Schmidt number, ratio parameter, reaction rate parameter, temperature ratio parameter, activation energy constant, thermal relaxation parameter, solutal relaxation parameter, radiation parameter, heat generation coefficient, porosity parameter, Forchheimer number, mixed convection parameter, Buoyancy ratio parameter, nonlinear concentration convection parameter, nonlinear thermal convection parameter:

$$We = c_1 \Gamma(\text{Re})^{\frac{1}{2}}; M = \left( \frac{\sigma_f B_0^2}{\rho_f c_1} \right); \text{Pr} = \left( \frac{\mu c_p}{k} \right)_f; \text{Sc} = \frac{\nu_f}{D_f};$$

$$A = \frac{c_2}{c_1}; K_E = \frac{k_f^2}{c_1}; a_1 = \frac{T_w^* - T_\infty^*}{T_\infty^*};$$

$$E = \frac{E_a}{k_e T_\infty}; \delta_1 = \lambda_2 c_1; \delta = \lambda_1 c_1; \text{Rd} = \frac{4\sigma^* T_\infty^{*4}}{k_f k^*};$$

$$Q = \frac{Q_0}{c_1 \rho_f c_p}; K = \frac{\nu_{\text{tnf}}}{c_1 k}; F = \frac{c_b l}{\sqrt{k}}; L^* = \frac{G_1}{\text{Re}^2};$$

$$N = \frac{G_2}{G_1}; M^* = \frac{\gamma_2^*(C_w^* - C_\infty^*)}{\gamma_1^*}; G = \frac{\beta_2^*(T_w^* - T_\infty^*)}{\beta_1^*},$$

where  $G_1$  and  $G_2$  represents the thermal and concentration Buoyancy numbers given by  $G_1 = \frac{g \beta_1^*(T_w^* - T_\infty^*) x^3}{\nu^2}$ ,  $G_2 = \frac{g \beta_1^*(C_w^* - C_\infty^*) x^3}{\nu^2}$ .

Fluid flow rate is also influenced by drag coefficient, thermal transfer ratio, and mass transfer ratio, which are calculated by:

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho U_w^2}, \quad \text{Nu} = \frac{x q_w}{k_f (T_w^* - T_\infty^*)}, \quad \text{Sh} = \frac{x m_w}{D_f (C_w^* - C_\infty^*)},$$

where nondimensional form of wall shear stress ( $\tau_w$ ), wall heat flux ( $q_w$ ), and wall mass flux ( $m_w$ ) are

$$\tau_w|_{y=0} = \left[ \frac{\mu_{\text{tnf}}}{\mu_f} \frac{u_y}{1 + \Gamma(u_y)^m} \right]_{y=0},$$

$$q_w|_{y=0} = - \frac{k_{\text{tnf}}}{k_f} (T_y^*)|_{y=0},$$

$$m_w|_{y=0} = - \frac{D_{\text{tnf}}}{D_f} (C_y^*)|_{y=0}.$$

Now using these dimensionless form, the expression for Skin friction, Nusselt number, Sherwood number

becomes  $\text{Re}^{\frac{1}{2}} C_f = \left[ \frac{\mu_{\text{tnf}}}{\mu_f} \frac{f''}{(1 + (\text{Wef}'')^m)^2} \right]$ ,  $\text{Re}^{\frac{1}{2}} \text{Nu} = - \left[ \frac{k_{\text{tnf}}}{k_f} + \frac{4}{3} \text{Rd} \right]$

$\theta'(0)$ , and  $\text{Re}^{\frac{1}{2}} \text{Sh} = - \frac{D_{\text{tnf}}}{D_f} \phi'(0)$  (Table 1).

### 3.1 First-order truncation

For this first-order truncation, assumption is that if  $\frac{\partial(\cdot)}{\partial \xi}$  ( $=0$ ), and  $\frac{\partial(\cdot)}{\partial \eta}$  ( $=0$ ) from equations (9)–(11) then the resulting equation becomes

$$\begin{aligned} & \frac{\nu_{\text{tnf}}}{\nu_f} (1 + (1 - m)(\text{Wef}'')^m) \frac{\partial^3 f}{\partial \eta^3} \\ & + \left\{ f \frac{\partial^2 f}{\partial \eta^2} - \left( \frac{\partial f}{\partial \eta} \right)^2 - \frac{\sigma_{\text{tnf}}}{\sigma_f} \frac{\rho_f}{\rho_{\text{tnf}}} \left[ A^2 + \left( A - M \frac{\partial f}{\partial \eta} \right) \right] \right. \\ & \left. - K \frac{\partial f}{\partial \eta} - F \xi \left( \frac{\partial f}{\partial \eta} \right)^2 + L^* \theta (1 + G \theta) + N \phi (1 + M^* \phi) \right\} \\ & \times (1 + (\text{Wef}'')^m)^2 = 0, \end{aligned} \quad (13)$$

$$\begin{aligned} & \left\{ \frac{k_{\text{tnf}}}{k_f} + \frac{4}{3} \text{Rd} \right\} \theta'' - \delta \text{Pr} f^2 \theta'' - \delta \text{Pr} f f' \theta' + \text{Pr} f \theta' + Q \text{Pr} \theta \\ & = 0 \end{aligned} \quad (14)$$

**Table 1:** Thermophysical properties and their correlations

| Thermophysical properties | Correlations                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|---------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Density                   | $\rho_{mnf} = (1 - \varphi_1) \{ (1 - \varphi_2) [ (1 - \varphi_3) \rho_f + \varphi_3 \rho_3 ] + \varphi_2 \rho_2 \} + \varphi_1 \rho_1$                                                                                                                                                                                                                                                                                                                                                                 |
| Heat capacity             | $(\rho c_p)_{mnf} = (1 - \varphi_1) (1 - \varphi_2) [ (1 - \varphi_3) (\rho c_p)_f + \varphi_3 (\rho c_p)_3 ] + \varphi_2 (\rho c_p)_2 + \varphi_1 (\rho c_p)_1$                                                                                                                                                                                                                                                                                                                                         |
| Viscosity                 | $\frac{\mu_{mnf}}{\mu_f} = \frac{1}{(1 - \varphi_1)^{2.5} (1 - \varphi_2)^{2.5} (1 - \varphi_3)^{2.5}}$                                                                                                                                                                                                                                                                                                                                                                                                  |
| Electric conductivity     | $\frac{\sigma_{mnf}}{\sigma_f} = \frac{\sigma_1 + 2\sigma_{dnf} - 2\varphi_1(\sigma_{dnf} - \sigma_1)}{\sigma_1 + 2\sigma_{dnf} + \varphi_1(\sigma_{dnf} - \sigma_1)}, \quad \frac{\sigma_{dnf}}{\sigma_f} = \frac{\sigma_2 + 2\sigma_{nf} - 2\varphi_2(\sigma_{nf} - \sigma_2)}{\sigma_2 + 2\sigma_{nf} + \varphi_2(\sigma_{nf} - \sigma_2)},$<br>$\frac{\sigma_{nf}}{\sigma_f} = \frac{\sigma_3 + 2\sigma_f - 2\varphi_3(\sigma_f - \sigma_3)}{\sigma_3 + 2\sigma_f + \varphi_3(\sigma_f - \sigma_3)}$ |
| Mass conductance          | $\frac{D_{mnf}}{D_f} = \frac{1}{(1 - \varphi_1)(1 - \varphi_2)(1 - \varphi_3)}$                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Thermal conductivity      | $\frac{k_{\text{tnf}}}{k_{dnf}} = \frac{k_1 + 2k_{dnf} - 2\varphi_1(k_{dnf} - k_1)}{k_1 + 2k_{dnf} + \varphi_1(k_{dnf} - k_1)}, \quad \frac{k_{dnf}}{k_{nf}} = \frac{k_2 + 2k_{nf} - 2\varphi_2(k_{nf} - k_2)}{k_2 + 2k_{nf} + \varphi_2(k_{nf} - k_2)},$<br>$\frac{k_{nf}}{k_f} = \frac{k_3 + 2k_f - 2\varphi_3(k_f - k_3)}{k_3 + 2k_f + \varphi_3(k_f - k_3)}$                                                                                                                                         |

$$\begin{aligned} \frac{D_{\text{tnf}}}{D_f} \phi'' - \delta_1 \text{Sc} f^2 \phi'' - \delta_1 \text{Sc} f f' \phi' + \text{Sc} f \phi' \\ - K_E \text{Sc} (1 + \theta \alpha_1)^m \phi \exp\left(\frac{-E}{1 + \theta \alpha_1}\right) = 0 \end{aligned} \quad (15)$$

along with boundary conditions

$$\left\{ \begin{array}{l} \text{At } \eta = 0: \quad f(\xi, \eta) = 0, \quad f'(\xi, \eta) = 1, \quad \theta(\xi, \eta) = 1, \\ \phi(\xi, \eta) = 1 \\ \text{When } \eta \rightarrow \infty: \quad f'(\xi, \eta) \rightarrow A, \quad \theta(\xi, \eta) = \phi(\xi, \eta) \rightarrow 0 \end{array} \right\}.$$

### 3.2 Second-order truncation

For the second-order truncation, we make the following assumptions:

$$\frac{\partial^2 \theta}{\partial \xi \partial \eta} \equiv \frac{\partial q}{\partial \eta}, \quad \frac{\partial \phi}{\partial \xi} \equiv g, \quad \frac{\partial^2 \phi}{\partial \xi \partial \eta} \equiv \frac{\partial g}{\partial \eta},$$

and

$$\frac{\partial f}{\partial \xi} \equiv p, \quad \frac{\partial^2 f}{\partial \xi \partial \eta} \equiv \frac{\partial p}{\partial \eta}, \quad \frac{\partial^3 f}{\partial \xi^2 \partial \eta} \equiv \frac{\partial^2 p}{\partial \eta^2}, \quad \frac{\partial \theta}{\partial \xi} \equiv q,$$

as applied in equations (9)–(12).

$$\begin{aligned} \frac{\nu_{\text{tnf}}}{\nu_f} \left[ 1 + (1 - m) \left( \text{We} \frac{\partial^2 f}{\partial \eta^2} \right)^m \right] \frac{\partial^3 f}{\partial \eta^3} + \left\{ f \frac{\partial^2 f}{\partial \eta^2} + \xi \left[ p \frac{\partial^2 f}{\partial \eta^2} \right. \right. \\ \left. \left. - \frac{\partial f}{\partial \eta} \frac{\partial p}{\partial \eta} \right] - \left( \frac{\partial f}{\partial \eta} \right)^2 - \frac{\sigma_{\text{tnf}} \rho_f}{\sigma_f \rho_{\text{tnf}}} [A^2 \right. \\ \left. + \left( A - M \frac{\partial f}{\partial \eta} \right) - K \frac{\partial f}{\partial \eta} - F \xi \left( \frac{\partial f}{\partial \eta} \right)^2 \right. \\ \left. + L^* \theta (1 + G\theta) + N \phi (1 + M^* \phi) \right\} \\ \times \left[ 1 + \left( \text{We} \frac{\partial^2 f}{\partial \eta^2} \right)^m \right] = 0, \end{aligned} \quad (16)$$

$$\begin{aligned} \left( \frac{k_{\text{tnf}}}{k_f} + \frac{4}{3} \text{Rd} \right) \frac{\partial^2 \theta}{\partial \eta^2} - \delta \text{Pr} (f^2 + 2f\xi p + \xi^2 p^2) \frac{\partial^2 \theta}{\partial \eta^2} \\ + \delta \text{Pr} \left[ 2\xi p \frac{\partial f}{\partial \eta} + \xi^2 \frac{\partial f}{\partial \eta} \frac{\partial p}{\partial \xi} - f \frac{\partial f}{\partial \eta} - \xi f \frac{\partial p}{\partial \eta} - \xi p \frac{\partial f}{\partial \eta} \right. \\ \left. - \xi^2 p \frac{\partial p}{\partial \eta} \right] \frac{\partial \theta}{\partial \eta} - \text{Pr} \left[ \xi \left( q \frac{\partial f}{\partial \eta} - p \frac{\partial \theta}{\partial \eta} \right) - f \frac{\partial \theta}{\partial \eta} \right] \\ + \delta \text{Pr} \left[ 2\xi f \frac{\partial f}{\partial \eta} \frac{\partial q}{\partial \eta} + 2\xi^2 p \frac{\partial f}{\partial \eta} \frac{\partial q}{\partial \eta} + \xi^2 p q \frac{\partial^2 f}{\partial \eta^2} \right. \\ \left. + f \xi p \frac{\partial^2 f}{\partial \eta^2} - \xi^2 q \frac{\partial f}{\partial \eta} \frac{\partial p}{\partial \eta} - \xi q \left( \frac{\partial f}{\partial \eta} \right)^2 \right] + Q \text{Pr} \theta = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{D_{\text{tnf}}}{D_f} \frac{\partial^2 \phi}{\partial \eta^2} - \delta_1 \text{Sc} (f^2 + 2f\xi p + \xi^2 p^2) \frac{\partial^2 \phi}{\partial \eta^2} \\ + \delta_1 \text{Sc} \left[ 2\xi p \frac{\partial f}{\partial \eta} + \xi^2 \frac{\partial f}{\partial \eta} \frac{\partial p}{\partial \xi} - f \frac{\partial f}{\partial \eta} - \xi f \frac{\partial p}{\partial \eta} - \xi p \frac{\partial f}{\partial \eta} \right. \\ \left. - \xi^2 p \frac{\partial p}{\partial \eta} \right] \frac{\partial \phi}{\partial \eta} - \text{Sc} \left[ \xi \left( g \frac{\partial f}{\partial \eta} - p \frac{\partial \phi}{\partial \eta} \right) - f \frac{\partial \phi}{\partial \eta} \right] \\ + \delta_1 \text{Sc} \left[ 2\xi f \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \eta} + 2\xi^2 p \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \eta} + \xi^2 p g \frac{\partial \phi}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} \right. \\ \left. + f \xi p \frac{\partial^2 f}{\partial \eta^2} - \xi^2 g \frac{\partial f}{\partial \eta} \frac{\partial p}{\partial \eta} - \xi p \left( \frac{\partial f}{\partial \eta} \right)^2 \right] \\ - K_E \text{Sc} (1 + \theta \alpha_1)^m \phi \exp\left(\frac{-E}{1 + \theta \alpha_1}\right) = 0, \end{aligned} \quad (18)$$

$$\begin{aligned} \text{At } \eta = 0: \quad f(\xi, \eta) = -\xi p, \quad \frac{\partial f(\xi, \eta)}{\partial \eta} = 1, \\ \theta(\xi, \eta) = 1, \quad \phi(\xi, \eta) = 1 \\ \text{When } \eta \rightarrow \infty: \quad \frac{\partial f(\xi, \eta)}{\partial \eta} \rightarrow A, \\ \theta(\xi, \eta) = \phi(\xi, \eta) \rightarrow 0, \end{aligned} \quad (19)$$

where  $p$ ,  $q$ , and  $g$  are unknowns. To find these, we have to differentiate equations (16)–(19) w.r.t the parameter  $\xi$  and the terms such as  $\frac{\partial p}{\partial \xi}$ ,  $\frac{\partial g}{\partial \xi}$ ,  $\frac{\partial q}{\partial \xi}$ , and  $\frac{\partial^2 q}{\partial \xi \partial \eta}$ ,  $\frac{\partial^2 g}{\partial \xi \partial \eta}$ ,  $\frac{\partial^2 p}{\partial \xi \partial \eta}$ ,  $\frac{\partial^3 p}{\partial \xi \partial \eta^2}$ , are eliminated and derivatives w.r.t.  $\eta$  are denoted by primes (').

$$\begin{aligned} (3f' - 2pf'' - fp'' + \xi(p'^2 - pp''))(1 + (\text{Wef}'')^m)^2 \\ + 2m\text{Wep}''(\text{Wef}'')^{m-1}(1 + (\text{Wef}'')^m) \\ \times [f''^2 - ff'' + \xi(f'p' - pf'')] \\ = \frac{\nu_{\text{tnf}}}{\nu_f} [(1 + (1 - m)(\text{Wef}'')^m)p'' \\ + m(1 - m)\text{Wep}''(\text{wef}'')^{m-1}f'''] \\ + (1 + (\text{Wef}'')^m)^2 \left[ \frac{\sigma_{\text{tnf}} \rho_f}{\sigma_f \rho_{\text{tnf}}} Mp' - Kp' - Ff'^2 \right. \\ \left. + 2\xi Ff'p' + L^* \theta Gq \right. \\ \left. + L^*(1 + G\theta)q + M^* N \phi g + N(1 + M^* \phi)g \right] \\ + 2m\text{Wep}''(\text{Wef}'')^{m-1}(1 + (\text{Wef}'')^m) \\ \times \left[ -\frac{\sigma_{\text{tnf}} \rho_f}{\sigma_f \rho_{\text{tnf}}} [A^2 + \left( A - M \frac{\partial f}{\partial \eta} \right) \right] \\ - Kf' - F\xi f''^2 + L^* \theta (1 + G\theta) \\ + N \phi (1 + M^* \phi)(1 + (\text{Wef}'')^m)^2, \end{aligned} \quad (20)$$

$$\begin{aligned}
& \text{Pr}(f'q - 2p\theta'\xi(p'q - pq') - fq') \\
& + \text{Pr}\delta(qf'^2 + 4\xi qf'p' + \xi^2 p'^2 q - qff'' - 3\xi pqf'' \\
& - \xi fqp'' \\
& - \xi^2 pqp'' - 8\xi pf'q' - ff'q' + 2f\theta'p' \\
& + \xi pp'\theta' - f\xi p'q' - \xi^2 pp'q' \\
& + 4fp\theta'' + f^2 q'' + 4\xi p^2 \theta'' + \xi^2 p^2 q'' + 2f\xi pq''),
\end{aligned} \quad (21)$$

$$\begin{aligned}
& \text{Sc}(f'g - 2p\phi' + \xi(p'g - pg') - fg') \\
& + \text{Sc}\delta_1(gf'^2 + 4\xi gf'p' + \xi^2 p'^2 g - gff'' - 3\xi pgf'' \\
& - \xi fgp'' \\
& - \xi^2 pgp'' - 8\xi pf'g' - ff'g' + 2f\phi'p' \\
& + \xi pp'\phi' - f\xi p'g' - \xi^2 pp'g' + 4fp\phi'' \\
& + f^2 g'' + 4\xi p^2 \phi'' + \xi^2 p^2 g'' + 2f\xi pg''),
\end{aligned} \quad (22)$$

with boundary conditions

$$\begin{aligned}
& \text{At } \eta = 0 : p(\xi, \eta) = 0, \quad p'(\xi, \eta) = 0, \quad q(\xi, \eta) = 0, \\
& g(\xi, \eta) = 0 \\
& \text{When } \eta \rightarrow \infty : p'(\xi, \eta) \rightarrow 0, \quad q(\xi, \eta) = g(\xi, \eta) \rightarrow 0.
\end{aligned} \quad (23)$$

## 4 Numerical methodology

The nondimensionalized governing equations (20)–(22) are nonlinear coupled equations with boundary conditions (23) solved computationally using MATLAB's built-in `bvp4c` solver, which is designed to handle boundary value problems (BVPs) efficiently. This solver incorporates the Lobatto IIIA method, a finite-difference algorithm that employs the three-stage Lobatto IIIA formula to achieve a

fourth-order accurate continuous solution across the integration domain. The interface of `bvp4c` is intuitive, enabling users to define the differential equations, boundary conditions, and initial guesses with relative ease, making it accessible even with basic programming skills.

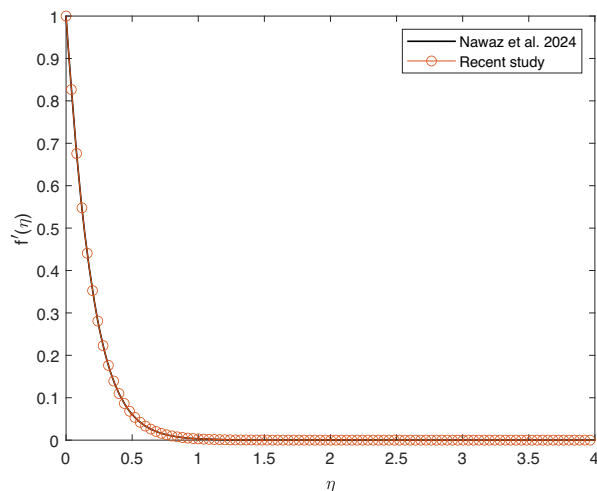
To address the nonlinear coupled differential equations, the system is transformed into a set of first-order ordinary differential equations (ODEs) through suitable variable substitutions. This reformulation is performed after applying a non-similarity transformation to the original partial differential equations (PDEs), which reduces them to a system of dimensionless ODEs for computational simplicity and accuracy.

The `bvp4c` solver requires an initial guess for the solution, as BVPs can have multiple possible solutions. It also demands an initial mesh to approximate the behavior of the solution. The algorithm refines the mesh iteratively, optimizing the solution while minimizing computational effort. A robust error control mechanism is implemented within `bvp4c` [61] ensuring that the solution converges to a predefined tolerance, typically  $10^{-6}$ . This iterative process continues until the error criterion is satisfied, providing a reliable and precise solution.

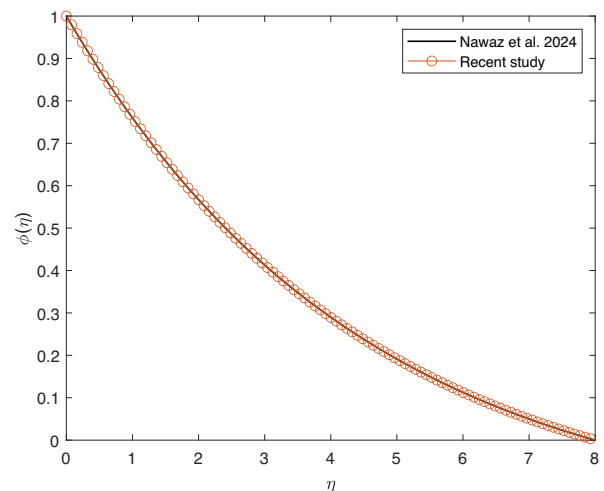
The system of first-order differential equation is given by

$$\begin{cases} f = y_1, & f' = y_2, & f'' = y_3, \\ p = y_4, & p' = y_5, & p'' = y_6, \\ \theta = y_7, & \theta' = y_8, \\ q = y_9, & q' = y_{10}, \\ \phi = y_{11}, & \phi' = y_{12}, \\ g = y_{13}, & g' = y_{14}. \end{cases}$$

Now boundary condition for the above system will be



(a)



(b)

**Figure 3:** Validation with existing: (a) velocity validation with existing one and (b) concentration validation with existing one.

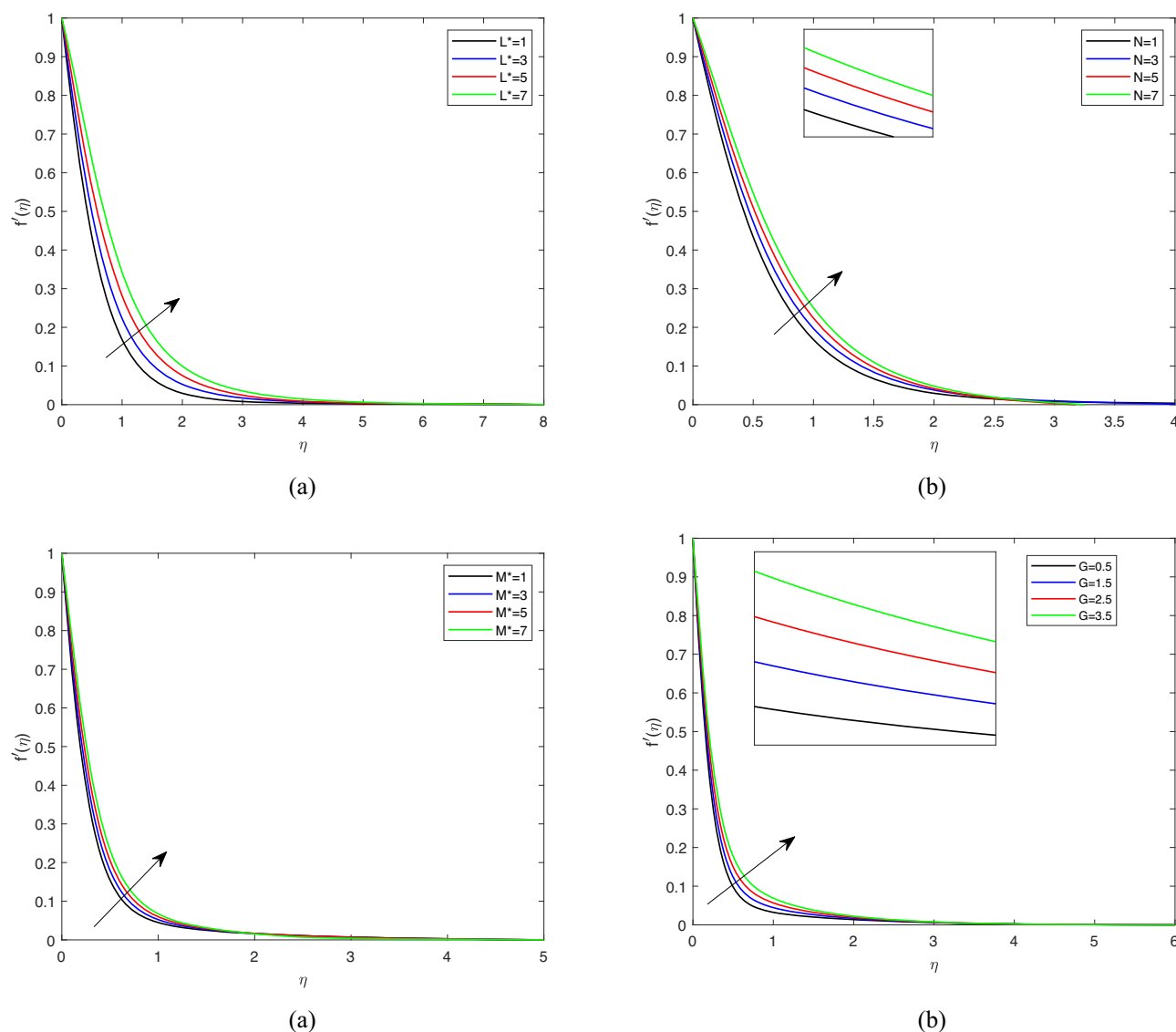
$$\left\{ \begin{array}{l} y_1(0) = 0; \quad y_2(0) = 1; \quad y_4(0) = 0; \quad y_5(0) = 0; \\ y_7(0) = 1; \quad y_9(0) = 0; \quad y_{11}(0) = 1; \quad y_{13}(0) = 0; \\ y_2(\infty) \rightarrow A; \quad y_4(\infty) \rightarrow 0; \quad y_5(\infty) \rightarrow 0; \\ y_6(\infty) \rightarrow 0; \quad y_7(\infty) \rightarrow 0; \quad y_9(\infty) \rightarrow 0; \\ y_{11}(\infty) \rightarrow 0; \quad y_{13}(\infty) \rightarrow 0; \end{array} \right\}.$$

## 5 Graphical results and discussion

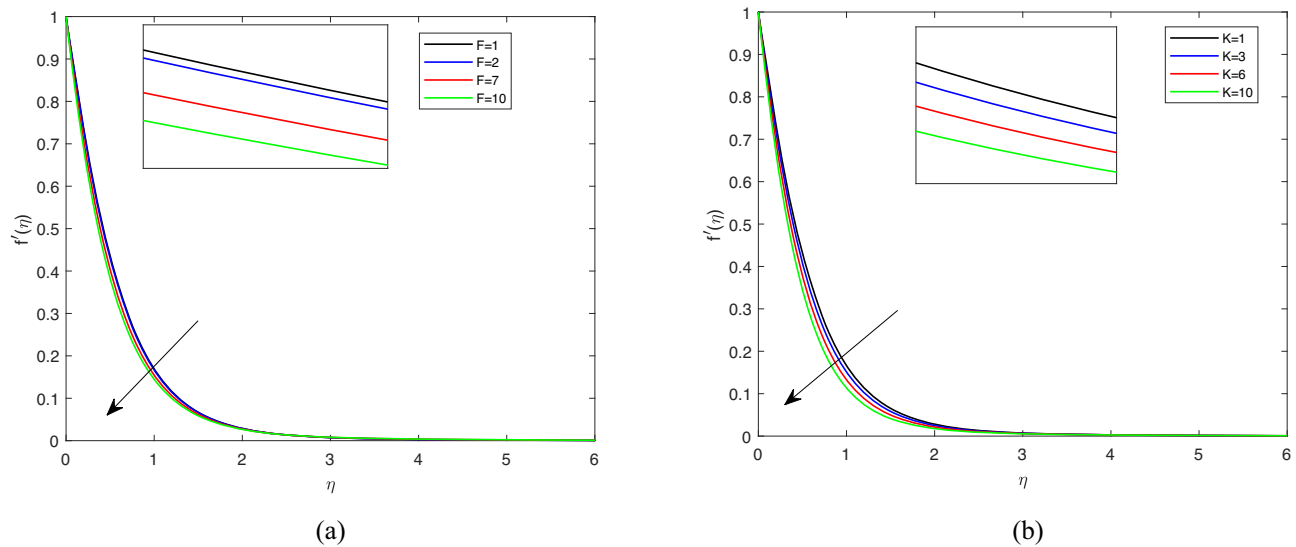
Graphical analyses present the changes in temperature, velocity, and concentration profiles, while both graphical

and tabular data reveal trends in skin friction, Sherwood number, and Nusselt number, emphasizing the effects of various key parameters.

To validate our analysis, results are compared with the findings of Nawaz *et al.* [56], setting specific parameters of our study to zero. Figure 3(a) and (b) displays a strong correlation with existing research. The plots illustrating the validation of velocity and concentration profiles, presented in Figure 3, align well with established studies, confirming the accuracy of our results.



**Figure 4:** Velocity distribution curve ( $K = 0.5, F = 0.6, L^* = 1, G = 0.5, N = 1, M^* = 0.5, Q = 1, Rd = 0.8, Ke = 1, E = 0.9, d = 1, Pr = 2.4, We = 1.8, Sc = 1.5, Ha = 1.3, \delta = 0.01, \delta_1 = 0.03, m = 0.02$ ). (a) Velocity distribution curve via factor ( $L^*$ ). Higher values of  $L^*$  increase velocity, (b) Velocity distribution curve via factor ( $N$ ). The velocity increases with an increasing value of  $N$ , (c) velocity distribution curve via factor ( $M^*$ ). Shows an upward trend in velocity with higher  $M^*$ , and (d) velocity distribution curve via factor ( $G$ ). As  $G$  increases, velocity rises, reflecting stronger thermal convection.



**Figure 5:** Velocity distribution curves *via*  $F$  and  $K$ . (a) Velocity distribution curve *via* factor ( $F$ ). The graph shows that increasing  $F$  leads to a decrease in velocity and (b) velocity distribution curve *via* factor ( $K$ ). The velocity decreases with an increase in  $K$ .

## 5.1 Observation for velocity profiles

The velocity profiles are highly responsive to influence the various parameters. The velocity distribution curves provide a visual representation of how velocity varies within the fluid domain.

The velocity distribution curves make it easier to interpret and understand the location of maximum velocity, boundary layer thickness, and complex flow patterns. Figure 4(a)–(d) illustrates how various factors, such as  $L^*$ ,  $N$ ,  $M$ , and  $G$ , significantly impact the velocity profiles. In the nondimensional form of the momentum equation, the parameter  $L^*$  represents the mixed convection effect. Since mixed convection is inversely related to the Reynolds number, fluid motion rises with decrease in the Reynolds number. This is evident in the velocity profile, where a higher value of  $L^*$  leads to increased fluid flow. The parameter  $K$  denotes the porosity factor, and as  $K$  increases, fluid velocity decreases because of the reduction in porosity permeability. Similarly, the Forchheimer number, which is inversely related to porosity permeability but directly dependent on the drag coefficient, shows a similar trend. For both parameters  $K$  and  $F$ , fluid motion decreases as permeability increases, as depicted in Figure 5(a) and (b). Additionally, nonlinear concentration convection ( $M$ ) and nonlinear thermal convection ( $G$ ) show that increasing their values leads to higher fluid velocity, producing similar effects on the velocity profile.

## 5.2 Observation for thermal profiles

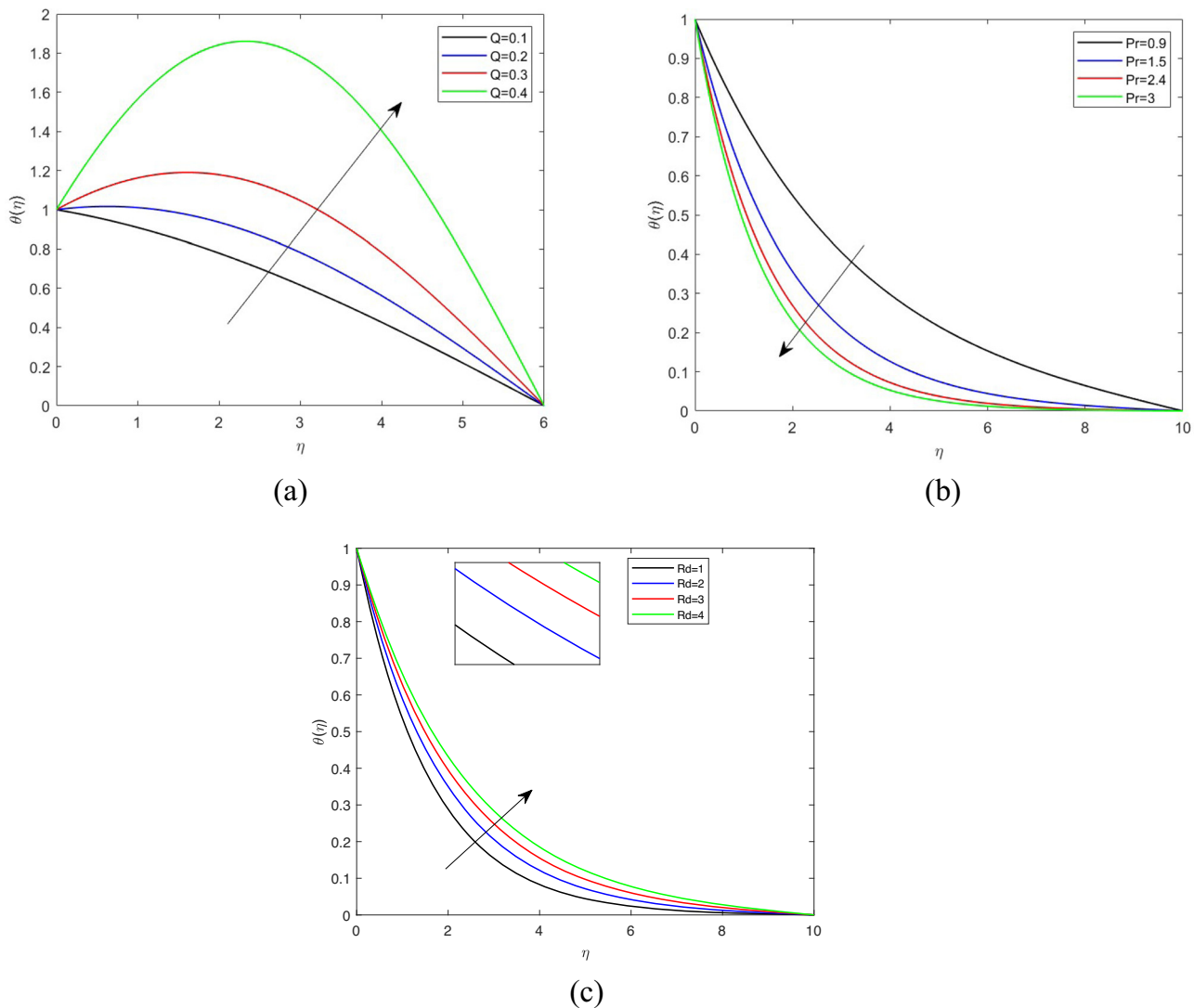
Thermal profiles provide crucial data for the design and innovation of new electronic components and systems. Different factors have a significant effect on thermal distribution patterns, as shown in Figure 6(a)–(c). The thermal distribution curve for various factors is depicted clearly, allowing for the analysis of heat behavior under varying conditions. The influence of the radiation factor ( $R_d$ ) on the heat distribution pattern is presented in Figure 6(c), showing that as the radiation parameter increases, so does the temperature. This demonstrates that higher radiation levels contribute to elevated temperature fields. Similarly, the influence of the heat source parameter ( $Q$ ) on the thermal distribution curve ( $\theta$ ) is shown in Figure 6(a). It is observed that increasing the heat source parameter results in a rise in temperature within the nanofluid. Furthermore, Figure 6(b) illustrates the effect of the Prandtl number ( $Pr$ ) on temperature. Since the Prandtl number is determined by the combination of viscosity, specific heat, and thermal conductivity, it has a significant impact on the thickness of thermal and velocity boundary layers. The results show that fluids with smaller Prandtl numbers (which have higher thermal conductivities) allow for faster heat diffusion within a thicker thermal boundary layer, while larger Prandtl number fluids exhibit thinner thermal boundary regions. Therefore, adjusting the Prandtl number can enhance the cooling rate in fluids that generate flow.

### 5.3 Observation for concentration profiles

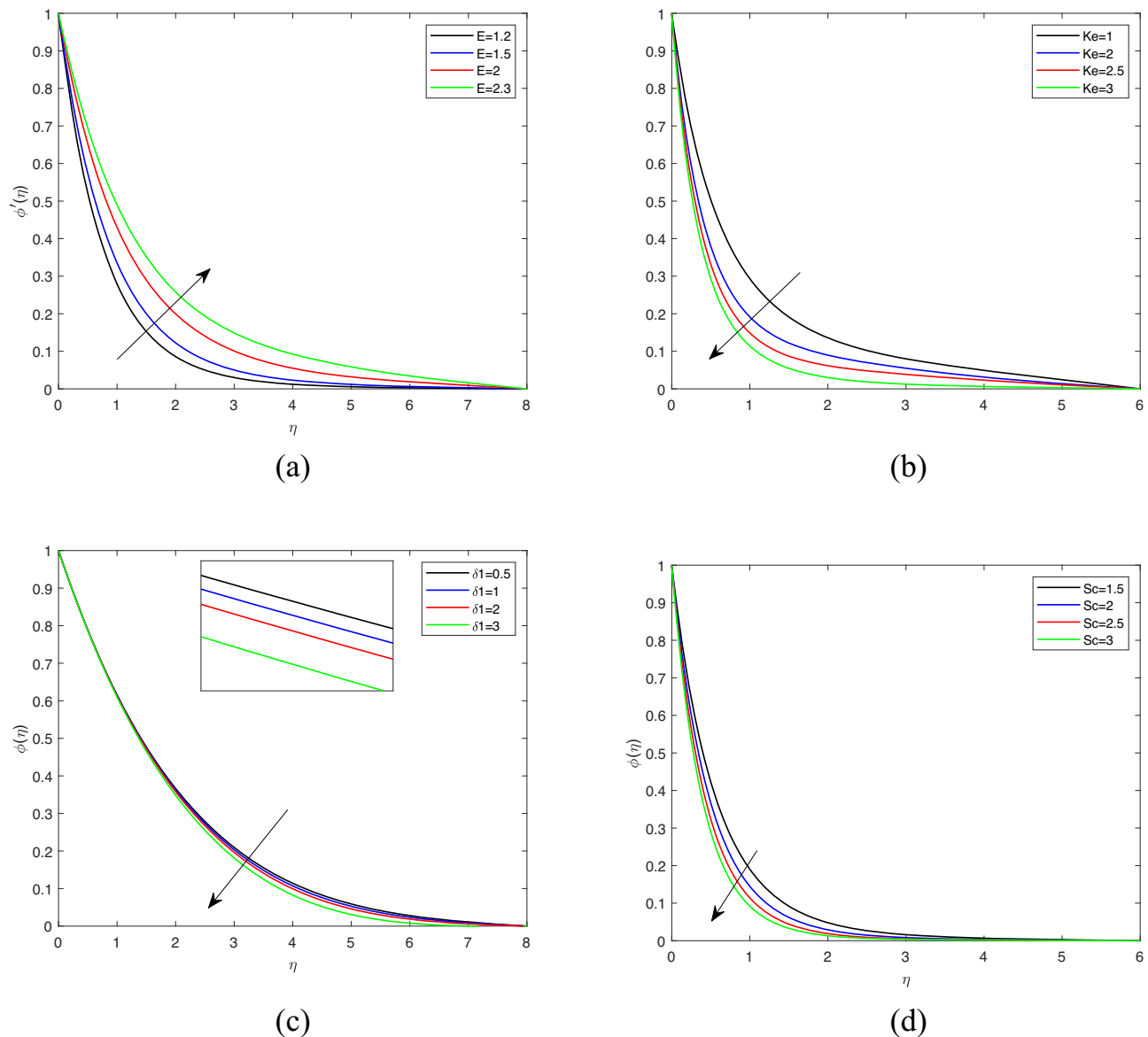
Concentration profiles offer an understanding of how solutes are transported within fluid flow, involving both diffusion and convection, which are key for developing accurate models of mass transfer. The concentration profiles vary under different influencing factors. Recent results obtained for these factors are shown in Figure 7(a)–(d). Figure 7(d) depicts the influence of the Schmidt number on concentration behavior. An increase in the Schmidt number leads to a reduction in the concentration profile. Additionally, Figure 7(a) highlights the impact of activation energy ( $E$ ) on the concentration field, the

concentration profile increases with a rise in the value of activation energy ( $E$ ), as illustrated by the graph.

This increase in activation energy enhances the Arrhenius function. Conversely, as the rate of chemical reaction ( $Ke$ ) increases, the concentration profile significantly decreases, as depicted in Figure 7(b). High chemical reaction rates lead to the thickening of the solute boundary layer, resulting in diminished concentration levels. Figure 7(c) presents the relationship between concentration and the thermal relaxation parameter  $\delta_1$ . An increase in the thermal relaxation parameter causes a decline in the concentration profile  $\phi(\eta)$ , indicating that fluctuations in thermal relaxation significantly impact mass transfer dynamics.



**Figure 6:** Thermal variation profiles for parameters  $Q$ ,  $Pr$ , and  $Rd$  ( $K = 0.5$ ,  $F = 0.6$ ,  $L^* = 1$ ,  $G = 0.5$ ,  $N = 1$ ,  $M^* = 0.5$ ,  $Q = 1$ ,  $Rd = 0.8$ ,  $Ke = 1$ ,  $E = 0.9$ ,  $d = 1$ ,  $Pr = 2.4$ ,  $We = 1.8$ ,  $Sc = 1.5$ ,  $Ha = 1.3$ ,  $\delta = 0.01$ ,  $\delta_1 = 0.03$ ,  $m = 0.02$ ). (a) Thermal profile for heat source parameter ( $Q$ ). Temperature increases with the increase in thermal energy, (b) thermal profile for Prandtl number ( $Pr$ ). Higher  $Pr$  values show thinner thermal boundary layers, and (c) Thermal profile for Radiation parameter ( $Rd$ ). The temperature rises with  $Rd$ , shows the impact of thermal radiation in elevating the fluid temperature field.



**Figure 7:** Concentration variation profiles ( $K = 0.5$ ,  $F = 0.6$ ,  $L^* = 1$ ,  $G = 0.5$ ,  $N = 1$ ,  $M^* = 0.5$ ,  $Q = 1$ ,  $Rd = 0.8$ ,  $Ke = 1$ ,  $E = 0.9$ ,  $d = 1$ ,  $Pr = 2.4$ ,  $We = 1.8$ ,  $Sc = 1.5$ ,  $Ha = 1.3$ ,  $\delta = 0.01$ ,  $\delta_1 = 0.03$ ,  $m = 0.02$ ). (a) Concentration profile for activation energy ( $E$ ). Concentration increases with  $E$ , (b) concentration profile for chemical reaction ratio ( $Ke$ ). A higher  $Ke$  reduces concentration, (c) concentration profile for thermal relaxation time parameter ( $\delta_1$ ). Increasing  $\delta_1$  reduces concentration, and (d) concentration profile for Schmidt number ( $Sc$ ). Concentration decreases with higher  $Sc$ .

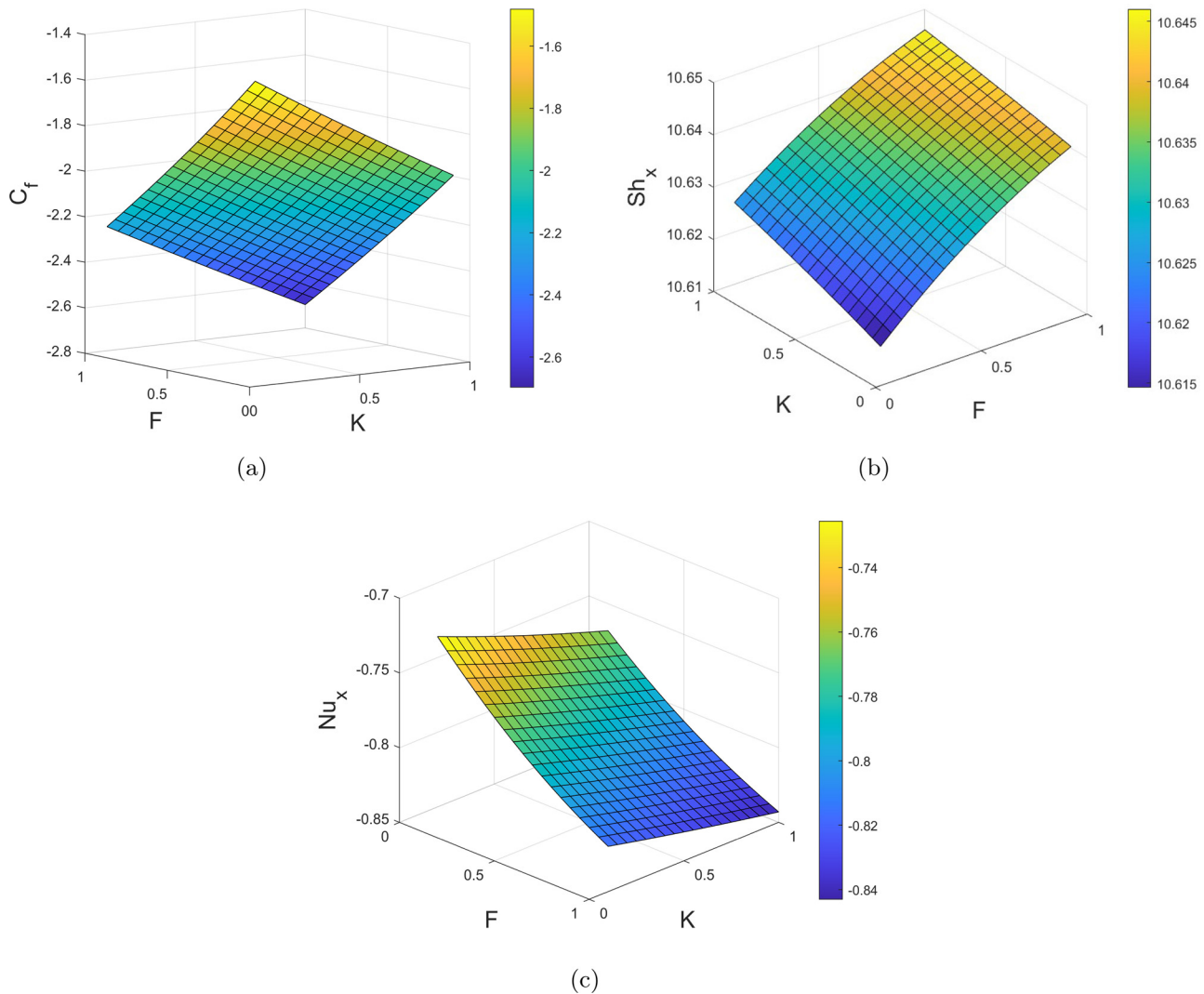
#### 5.4 Quantitative analysis of shear stress, thermal conductivity, and mass transfer efficiencies

The study of fluid flow, coupled with heat and mass transfer across various systems, depends significantly on specific dimensionless parameters. These parameters are essential because they offer valuable insights into the resistance to fluid movement, the efficiency of thermal energy transfer, and the effectiveness of mass diffusion across interfaces. A graphical and tabular attempt have been made to find the

variation in shear stress, heat transfer coefficient, and mass transfer coefficient for various parameters. Figure 8(a)–(c) and Figure 9(a)–(f) illustrate the graphical variations of the drag coefficient, Nusselt number, and Sherwood number.

**Impact of key parameters on shear stress:** Rising values of the Forchheimer number  $F$  and the porosity parameter  $K$  lead to a reduction in the Skin friction number.

Nonlinear concentration convection and nonlinear thermal convection parameters demonstrate an increase in the skin friction coefficient. Additionally, as the buoyancy ratio parameter increases, the skin friction coefficient also



**Figure 8:** Surface plot via  $C_f$ ,  $Sh$ ,  $Nu$ . (a) Skin-friction ( $C_f$ ), (b) Sherwood number ( $Sh$ ), and (c) Nusselt number ( $Nu$ ).

risers. This increase in skin friction typically leads to a reduction in the extent of the boundary layer. A summary of these results can be found in Tables 2–4 for further clarity.

#### Impact of key parameters on Nusselt number:

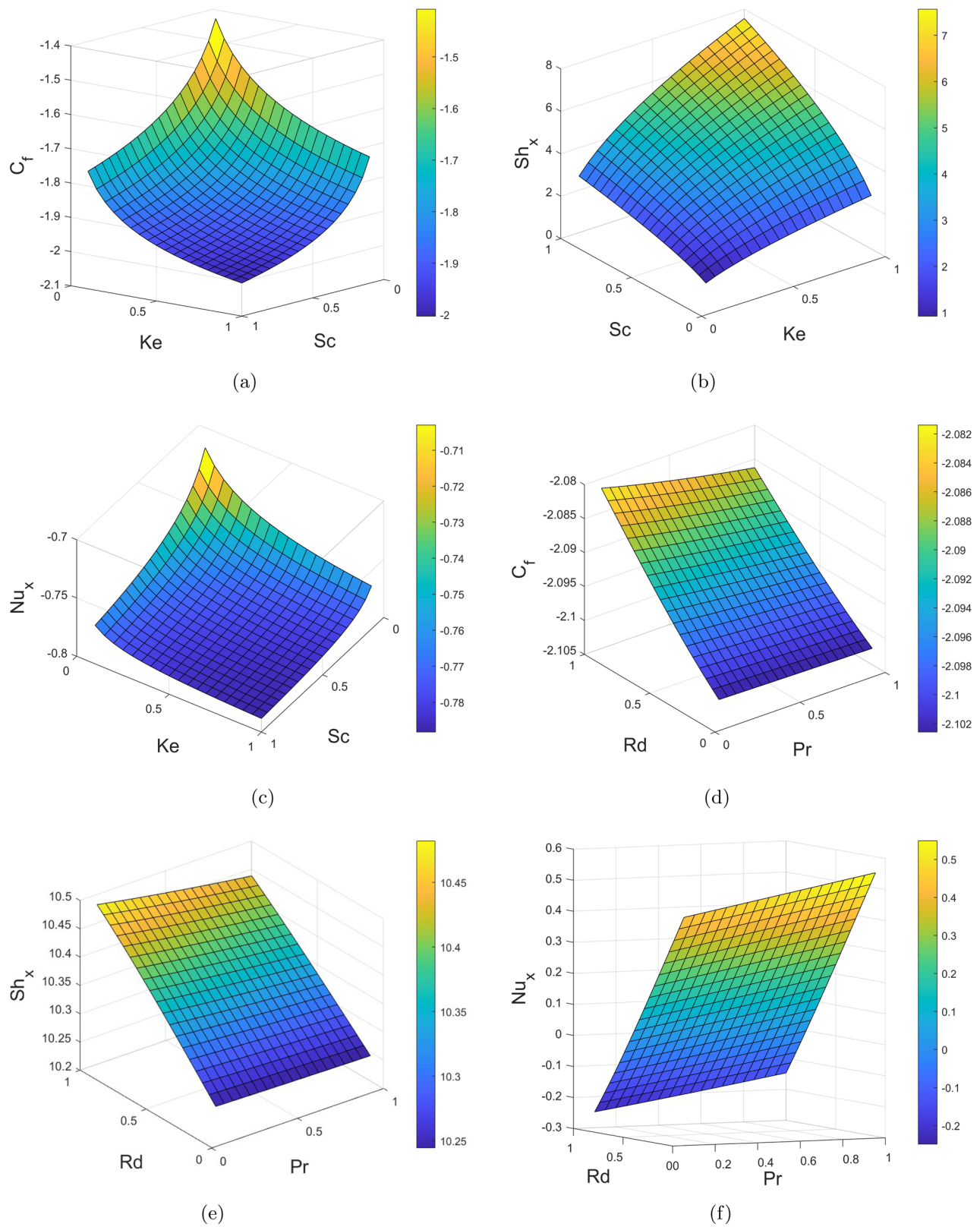
An enhancement in the radiation parameter  $R_d$  contributes to a rise in the Nusselt number (thermal conductivity), while a rise in the heat parameter  $Q$  generates a decline in the Nusselt number.

A higher Prandtl number leads to an increase in the Nusselt number. A higher Nusselt number typically signifies more effective convective heat transfer, which consequently leads to a thinner thermal boundary layer. This enhanced convective thermal transmission decreases the temperature gradient within the boundary layer, allowing

heat to transfer more quickly from the surface to the bulk fluid.

#### Impact of key parameters on Sherwood number:

The observations indicate that elevating both the Schmidt number and the chemical reaction parameter enhances the Sherwood number, signifying improved mass transfer to the boundary. As the activation energy parameter  $E$  increases, the Sherwood number decreases. Conversely, increasing the heat parameter  $Q$  leads to an enhancement of the Sherwood number. A higher Sherwood number is associated with a more compact concentration boundary layer. This increased convective mass transfer reduces the resistance to mass transfer, allowing the concentration of species to change more rapidly near the surface.



**Figure 9:** Surface plot for skin friction, Nusselt, and Sherwood numbers. (a) Skin-friction ( $C_f$ ), (b) Sherwood number ( $Sh$ ), (c) Nusselt number ( $Nu$ ), (d) Skin-friction ( $C_f$ ), (e) Sherwood number ( $Sh$ ), and (f) Nusselt number ( $Nu$ ).

**Table 2:** Comparison of skin friction, Nusselt number, and Sherwood number for tri-nanofluid

| $M$ | $Q$  | $F$ | $K$ | $Rd$ | $Pr$ | $We$ | $m$ | $Sc$ | $-(Re)^{\frac{1}{2}} Cf$ | $(Re)^{\frac{1}{2}} Nu$ | $(Re)^{\frac{1}{2}} Sh$ |
|-----|------|-----|-----|------|------|------|-----|------|--------------------------|-------------------------|-------------------------|
| 1   | 0.1  | 0.2 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 2.0029                   | 0.2421                  | 1.5793                  |
| 2   |      |     |     |      |      |      |     |      | 1.9756                   | 0.2374                  | 1.5808                  |
| 2.5 |      |     |     |      |      |      |     |      | 1.9479                   | 0.2351                  | 1.5824                  |
| 1   | 0.02 | 0.2 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 2.0038                   | 0.8793                  | 1.5483                  |
|     | 0.04 |     |     |      |      |      |     |      | 2.0036                   | 0.7324                  | 1.5554                  |
|     | 0.06 |     |     |      |      |      |     |      | 2.0034                   | 0.5778                  | 1.5629                  |
| 1   | 0.1  | 0.3 | 0.5 | 0.6  | 1.4  | 0.2  | 1   | 1    | 2.0045                   | 0.2422                  | 1.5792                  |
|     |      | 0.5 |     |      |      |      |     |      | 2.0092                   | 0.2423                  | 1.5792                  |
|     |      | 0.7 |     |      |      |      |     |      | 2.0137                   | 0.2425                  | 1.5791                  |
| 1   | 0.1  | 0.2 | 0.2 | 0.6  | 1.4  | 0.2  | 1   | 1    | 1.9928                   | 0.2413                  | 1.5796                  |
|     |      |     | 0.4 |      |      |      |     |      | 1.9996                   | 0.2419                  | 1.5794                  |
|     |      |     | 0.6 |      |      |      |     |      | 2.0056                   | 0.2429                  | 1.592                   |
| 1   | 0.1  | 0.2 | 0.5 | 0.1  | 1.4  | 0.2  | 1   | 1    | 2.0028                   | 0.0773                  | 1.5853                  |
|     |      |     |     | 0.2  |      |      |     |      | 2.0028                   | 0.1048                  | 1.5840                  |
|     |      |     |     | 0.3  |      |      |     |      | 2.0029                   | 0.1352                  | 1.5827                  |

**Table 3:** Comparison of skin friction, Nusselt number, and Sherwood number for hybrid-nanofluid

| $M$ | $Q$  | $F$ | $K$ | $Rd$ | $Pr$ | $We$ | $m$ | $Sc$ | $-(Re)^{\frac{1}{2}} Cf$ | $(Re)^{\frac{1}{2}} Nu$ | $(Re)^{\frac{1}{2}} Sh$ |
|-----|------|-----|-----|------|------|------|-----|------|--------------------------|-------------------------|-------------------------|
| 1   | 0.1  | 0.2 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 2.2384                   | 0.2921                  | 1.2674                  |
| 2   |      |     |     |      |      |      |     |      | 2.2118                   | 0.2897                  | 1.2681                  |
| 3   |      |     |     |      |      |      |     |      | 2.1931                   | 0.2872                  | 1.2689                  |
| 1   | 0.02 | 0.2 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 2.2311                   | 0.9186                  | 1.2461                  |
|     | 0.04 |     |     |      |      |      |     |      | 2.2311                   | 0.7742                  | 1.2510                  |
|     | 0.4  |     |     |      |      |      |     |      | 2.2385                   | 0.6223                  | 1.2561                  |
| 1   | 0.1  | 0.3 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 2.2399                   | 0.2922                  | 1.2674                  |
|     |      | 0.5 |     |      |      |      |     |      | 2.2429                   | 0.2922                  | 1.2673                  |
|     |      | 0.7 |     |      |      |      |     |      | 2.2458                   | 0.2923                  | 1.2673                  |
| 1   | 0.1  | 0.2 | 0.2 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 2.2287                   | 0.2919                  | 1.2675                  |
|     |      |     | 0.4 |      |      |      |     |      | 2.2362                   | 0.2920                  | 1.2674                  |
|     |      |     | 0.6 |      |      |      |     |      | 2.2406                   | 0.2922                  | 1.2673                  |
| 1   | 0.1  | 0.2 | 0.5 | 0.1  | 1.4  | 0.2  | 0.1 | 1    | 2.2383                   | 0.1131                  | 1.2713                  |
|     |      |     |     | 0.2  |      |      |     |      | 2.2384                   | 0.1439                  | 1.2704                  |
|     |      |     |     | 0.3  |      |      |     |      | 2.2384                   | 0.1773                  | 1.2696                  |

## 6 Comparative study of single-, hybrid- and tri-nanoparticle

Tri-nanoparticles are as follows:  $\phi_1 = 0.35$ ,  $\phi_2 = 0.25$ ,  $\phi_3 = 0.05$ , hybrid-nanoparticles are as follows:  $\phi_1 = 0.35$ ,  $\phi_2 = 0.25$ ,  $\phi_3 = 0$ , mono-nanoparticles are as follows:  $\phi_1 = 0.35$ ,  $\phi_2 = 0$ ,  $\phi_3 = 0$ .

The profiles shown in Figure 10(a)–(c) clearly compare the performance of tri-nanoparticle, di-nanoparticle, and mono-nanoparticle systems. For the comparison of tri-nanoparticle, di-nanoparticle, and mono-nanoparticle systems, parameters like the Forchheimer number, activation energy, and heat radiation were kept

at zero. This was done to focus only on the basic physical properties of the nanoparticles and their direct effects, ensuring a clear and fair comparison. The tri-nanoparticle system consistently performs better in terms of velocity, temperature, and concentration profiles, supporting the idea that using multiple nanoparticles provides added benefits. While the di-nanoparticle system is not as effective as the tri-nanoparticle system, it still shows improved performance over the mono-nanoparticle system across all parameters. These results are consistent with previous studies, highlighting the importance and impact of this research. Tri-nanoparticles have shown to significantly boost heat transfer efficiency in industrial cooling systems. This improvement results in faster

**Table 4:** Comparison of skin friction, Nusselt number, and Sherwood number for nanofluid

| $M$ | $Q$  | $F$ | $K$ | $Rd$ | $Pr$ | $We$ | $m$ | $Sc$ | $-(Re)^{\frac{1}{2}} Cf$ | $(Re)^{\frac{1}{2}} Nu$ | $(Re)^{\frac{1}{2}} Sh$ |
|-----|------|-----|-----|------|------|------|-----|------|--------------------------|-------------------------|-------------------------|
| 1   | 0.1  | 0.2 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 1.7101                   | 0.1372                  | 1.1086                  |
| 2   |      |     |     |      |      |      |     |      | 1.6914                   | 0.1344                  | 1.1095                  |
| 3   |      |     |     |      |      |      |     |      | 1.6772                   | 0.1330                  | 1.1099                  |
| 1   | 0.02 | 0.2 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 1.7066                   | 0.7699                  | 1.0901                  |
|     | 0.04 |     |     |      |      |      |     |      | 1.7066                   | 0.6260                  | 1.0943                  |
|     | 0.06 |     |     |      |      |      |     |      | 1.7102                   | 0.4733                  | 1.0988                  |
| 1   | 0.1  | 0.3 | 0.5 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 1.7116                   | 0.1372                  | 1.1086                  |
|     |      | 0.5 |     |      |      |      |     |      | 1.7146                   | 0.1373                  | 1.1086                  |
|     |      | 0.7 |     |      |      |      |     |      | 1.7177                   | 0.1374                  | 1.1085                  |
| 1   | 0.1  | 0.2 | 0.2 | 0.6  | 1.4  | 0.2  | 0.1 | 1    | 1.7033                   | 0.1368                  | 1.1088                  |
|     |      |     | 0.4 |      |      |      |     |      | 1.7078                   | 0.1371                  | 1.1087                  |
|     |      |     | 0.6 |      |      |      |     |      | 1.7124                   | 0.1373                  | 1.1086                  |
| 1   | 0.1  | 0.2 | 0.5 | 0.1  | 1.4  | 0.2  | 0.1 | 1    | 1.7101                   | 0.0028                  | 1.1126                  |
|     |      |     |     | 0.2  |      |      |     |      | 1.7101                   | 0.0240                  | 1.1117                  |
|     |      |     |     | 0.3  |      |      |     |      | 1.7101                   | 0.483                   | 1.1109                  |

cooling processes and lower energy usage, compared to mono and di-nanoparticles. In energy storage applications, such as batteries and supercapacitors, tri-nanoparticles enhance performance by providing higher energy densities and extended cycle life, surpassing the capabilities of mono- and di-nanoparticles. Tri-nanoparticles offer superior control over drug release rates in biomedical treatments. This leads to increased treatment effectiveness and fewer side effects, compared to mono- and di-nanoparticles.

## 7 Entropy generation analysis

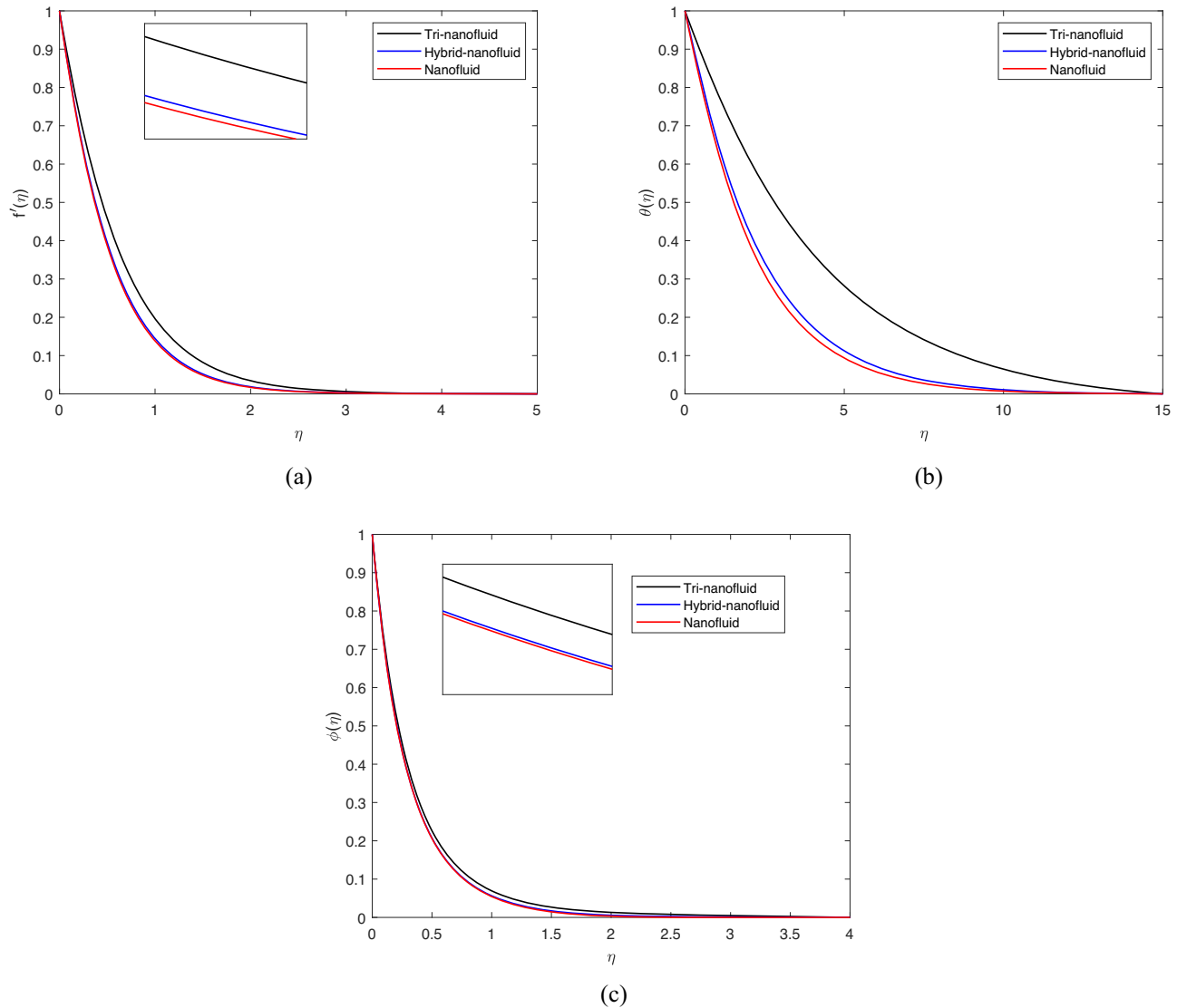
Entropy generation profiles give insights into how nanofluids behave when different nanoparticles are combined. This study emphasizes the potential to improve nanofluid formulations to achieve better performance, which is beneficial for applications like thermal management and various industrial processes. Analyzing entropy generation helps identify where energy losses happen in fluid systems, showing that not all heat is converted into work. Factors such as radiation effects on heat and mass transport, and the porous medium contribute to entropy generation. In this model, the expression for the rate of entropy generation [62] is as follows:

$$\begin{aligned}
 E_G = & \underbrace{\frac{k_f}{T_\infty^{*2}} \left[ \frac{k_{tnf}}{k_f} + \frac{16\sigma^* T_\infty^{*3}}{3k} \right] \left( \frac{\partial T^*}{\partial y} \right)^2}_{\text{Thermal radiation}} \\
 & + \underbrace{\frac{RD}{C_\infty^*} C_y^{*2} + \frac{RD}{T_\infty^*} C_y^* T_y^*}_{\text{Diffusive irreversibility}} \\
 & + \underbrace{\frac{1}{1 + (\Gamma u_y)^m \frac{\mu_{tnf}}{T_\infty^*} \left( \frac{\partial u}{\partial y} \right)^2}}_{\text{Fluid friction irreversibility}} \\
 & + \underbrace{\frac{\mu u^2}{T_\infty^* k}}_{\text{Porous medium}} + \underbrace{\frac{\sigma_{tnf} (U_e - u)^2 B_o^2}{\rho_{tnf}}}_{\text{Joule heating irreversibility}}
 \end{aligned} \quad (24)$$

Heat transfer-related entropy creation is represented by the first term in equation (24), diffusive irreversibility is represented by the second term, and porous medium-related entropy production is represented by the last term.

By using similarity transformation (7), nondimensional entropy generation rate becomes

$$\begin{aligned}
 N_G = & L \frac{\alpha_2}{\alpha_1} \phi'^2 + L \phi' \theta' + \left( 1 + \frac{4}{3} Rd \right) \alpha_1 \theta'^2 \\
 & + \frac{Br f'^2}{1 + (We f'')^m} + Br K f'^2 + M Br (A - f')^2,
 \end{aligned} \quad (25)$$



**Figure 10:** Comparison profiles of velocity, temperature, and concentration. (a) Velocity profile for comparison. Tri-nanofluid exhibits the highest velocity. (b) Temperature profile for comparison. Tri-nanofluid result in higher temperature profiles. (c) Concentration profile for comparison. The tri-nanoparticle system achieves the highest concentration profile.

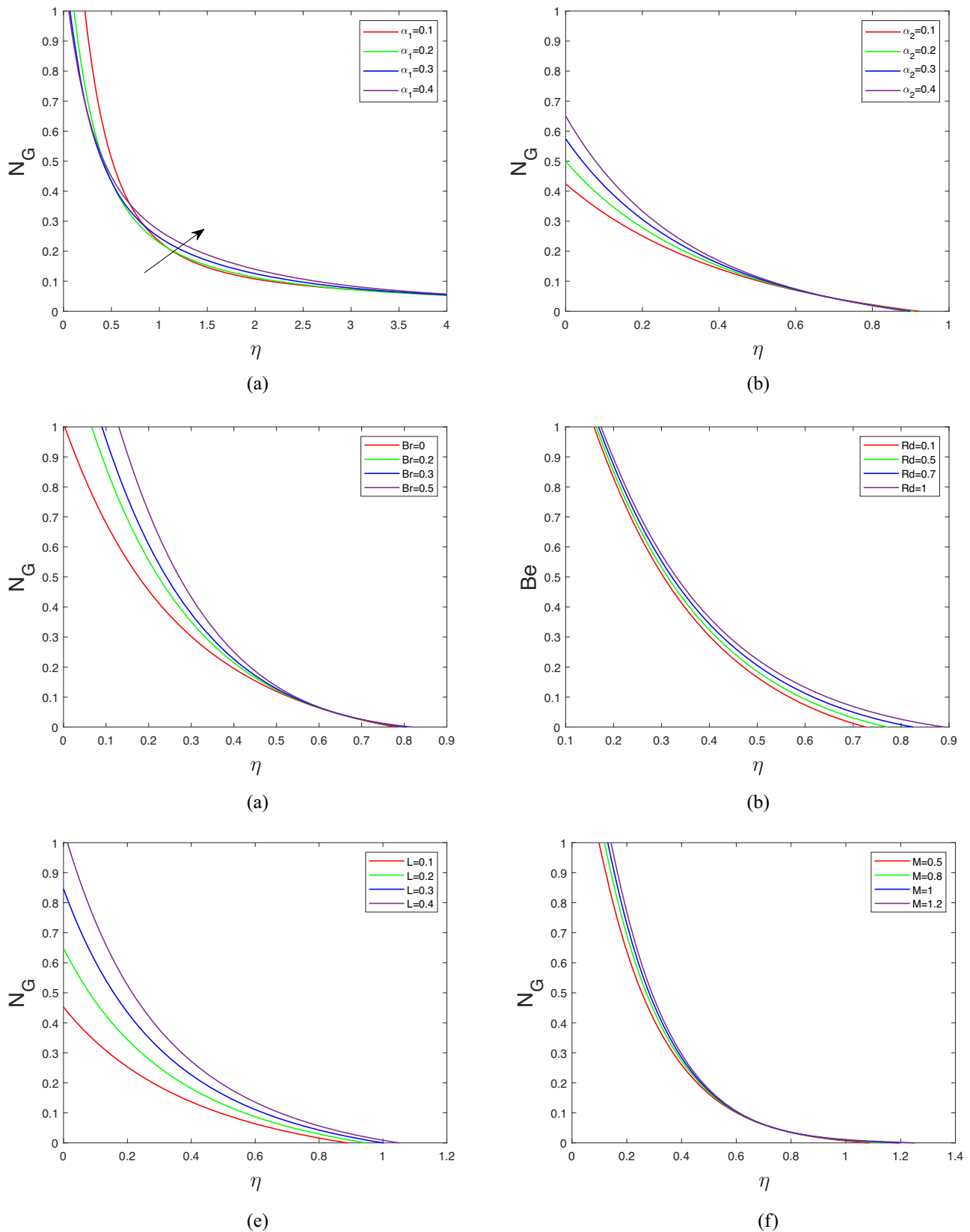
where  $L = \frac{RD(C_w^* - C_\infty^*)}{k_{\text{tnf}}}$  is the diffusion parameter,  $\alpha_1 = \frac{T_w^* - T_\infty^*}{T_\infty^*}$  and  $\alpha_2 = \frac{C_w^* - C_\infty^*}{C_\infty^*}$  represents temperature and concentration difference, respectively,  $M = \frac{\sigma_{\text{tnf}} B_0^2}{\rho_{\text{tnf}} c_1}$  represents magnetic parameter, and  $Br = \frac{\mu c_1^2 \xi^2 l^2}{k_{\text{tnf}} (T_w^* - T_\infty^*)}$  is the Brinkman number.

Bejan number is another dimensionless factor that evaluates the proportion of entropy generation attributed to heat transfer  $E_H$  relative to the total entropy generation  $E_G$  in a system. Mathematically it is written as,  $Be = \frac{E_H}{E_G}$ , i.e.,

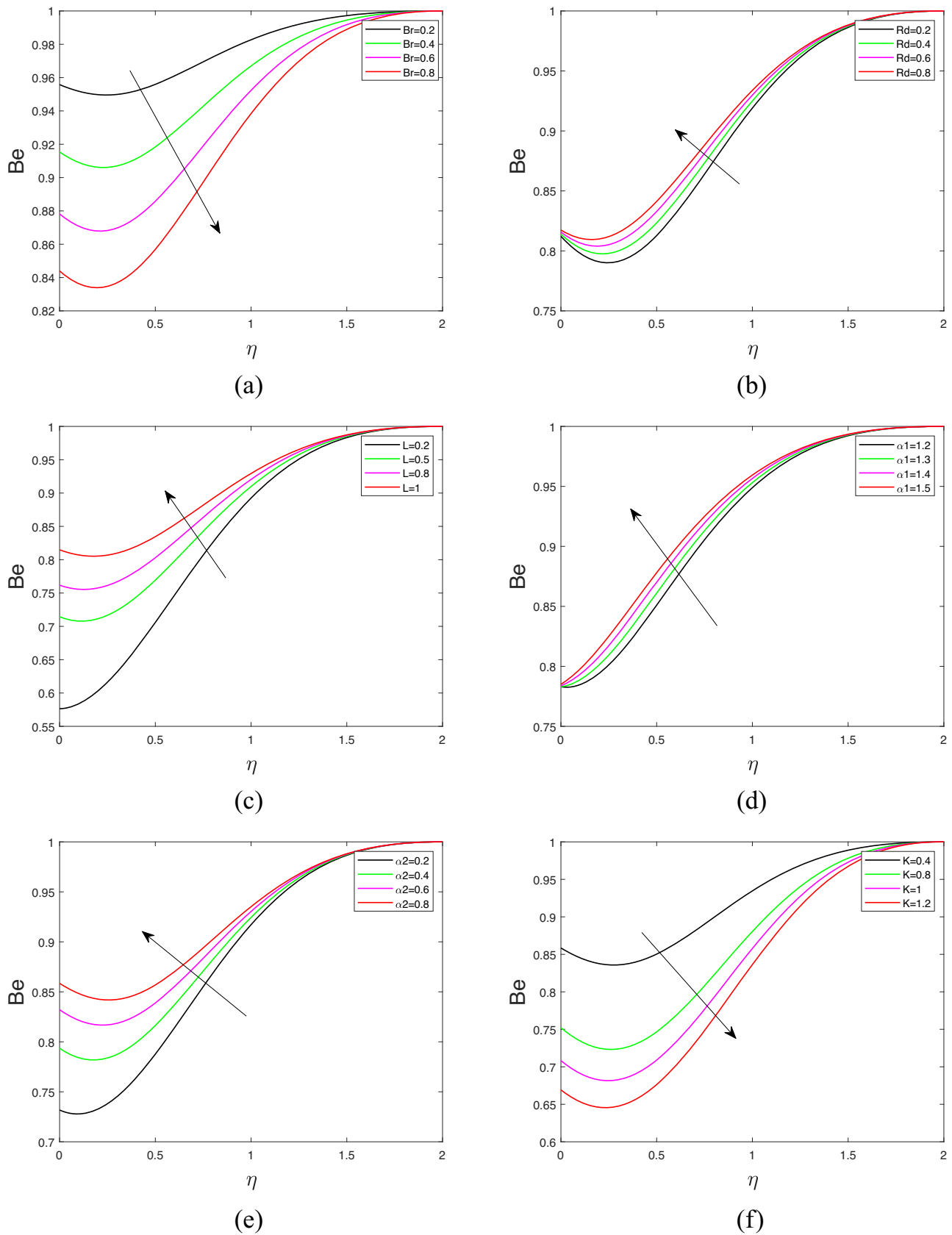
$$Be = \frac{\text{Heat and mass transfer irreversibility}}{\text{Total irreversibility}}$$

$$= \frac{L\phi'\theta' + \left(1 + \frac{4}{3}\text{Rd}\right)\alpha_1\theta'^2 + L\frac{\alpha_2}{\alpha_1}\phi'^2}{L\frac{\alpha_2}{\alpha_1}\phi'^2 + L\phi'\theta' + \left(1 + \frac{4}{3}\text{Rd}\right)\alpha_1\theta'^2 + \frac{\text{Br}f'^2}{1 + (\text{Wef}'')^m} + \text{Br}Kf'^2 + M\text{Br}(A - f')^2}$$

Generally Bejan number falls within the interval from 0 to 1. When the Bejan number is zero ( $Be = 0$ ), it indicates that heat transfer does not contribute to any irreversibility. A



**Figure 11:** Entropy generation results for different parameters ( $L = 0.6$ ,  $\alpha_1 = 0.6$ ,  $\alpha_2 = 0.5$ ,  $K = 0.6$ ,  $Rd = 0.6$ ,  $M = 1$ ,  $Br = 0.5$ ,  $Q = 0.5$ ,  $Ke = 1.5$ ,  $E = 1$ ,  $d = 1$ ). (a) Entropy generation for parameter ( $\alpha_1$ ). (b) Entropy generation for parameter ( $\alpha_2$ ), (c) entropy generation for Brinkman number (Br), (d) entropy generation for radiation parameter (Rd), (e) entropy generation for diffusion parameter (LII), and (f) entropy generation for magnetic parameter (M).



**Figure 12:** Bejan number *via* different influential parameters. ( $L = 1$ ;  $Rd = 0.2$ ;  $\alpha_1 = 0.6$ ;  $\alpha_2 = 0.5$ ;  $Br = 1$ ;  $K = 0.5$ ;  $M = 0.5$ ;  $Q = 0.5$ ;  $Ke = 1.5$ ;  $E = 1$ ;  $d = 1$ ). (a) Bejan no. variation for Brinkmann number ( $Br$ ), (b) Bejan no. variation for Radiation parameter ( $Rd$ ), (c) Bejan no. variation for parameter  $L$ , (d) Bejan no. variation for temperature difference ( $\alpha_1$ ), (e) Bejan no. variation for concentration difference ( $\alpha_2$ ), and (f) Bejan no. variation for parameter ( $K$ ).

**Table 5:** Physical properties of different materials

| Physical property | $\rho$ | $c_p$ | $\phi$ | $\sigma$             | $k$   |
|-------------------|--------|-------|--------|----------------------|-------|
| EG: water         | 1,050  | 3,288 | 0.0    | 0.00509              | 0.425 |
| Ag                | 10,500 | 235   | 0.35   | $6.3 \times 10^7$    | 492   |
| TiO <sub>2</sub>  | 4,250  | 686.2 | 0.25   | $2.4 \times 10^6$    | 8.953 |
| GO                | 1,800  | 717   | 0.05   | $1.1 \times 10^{-5}$ | 5,000 |

**Table 6:** Comparison assessment of friction drag for K and F is performed under the conditions when  $L^* = G = N = M = Q =$   
Rd = Ke = E = 0

| K   | F   | N.M. <i>et al.</i> [65] | Recent results |
|-----|-----|-------------------------|----------------|
| 0.2 | 0.2 | 1.930026                | 1.935934       |
| 0.4 | 0.5 | 2.321562                | 2.159261       |
| 0.8 | 0.7 | 2.478236                | 2.439254       |
| 1.5 | 1.2 | 2.934962                | 2.936234       |

Bejan number of one ( $Be = 1$ ) shows that the irreversibility from heat transfer equals the total system irreversibility. When the Bejan number is one-half ( $Be = 0.5$ ), it suggests that the total irreversibility is twice that caused by heat transfer alone. Understanding the Bejan number helps identify the primary causes of entropy generation. This allows for the optimization of thermal systems, reducing energy losses and improving efficiency.

## 7.1 Graphical evaluation of entropy generation and Bejan number

The impact of the parameters  $\alpha_1$ ,  $\alpha_2$ , Br, Rd,  $L$ , and  $M$  on entropy generation ( $N_G$ ) and Bejan number ( $Be$ ) is shown in Figures 11(a)–(f) and 12(a)–(f). Figure 11(a) illustrates the effect of the temperature difference parameter ( $\alpha_1$ ) on  $N_G$ . With an increase in the temperature variation, the value of  $N_G$  rises. Figure 11(b) demonstrates that  $N_G$  increases as the concentration difference parameter ( $\alpha_2$ ) rises. Specifically, Figure 11(c) highlights how  $N_G$  vary with respect to Br. It is clear from these figures that as Br increases,  $N_G$  also increases. Physically, Br represents the heat generated from fluid friction and molecular conduction. As Br increases, the system produces more heat, leading to greater disorder and, thus, a higher  $N_G$ . Alternatively, Figure 12(a) demonstrates that  $Be$  decreases with an increase in Br. Figure 12(d) illustrates the influence of the temperature differential parameter ( $\alpha_1$ )  $Be$ . With an

increase in the temperature variation, the values of  $Be$  also rise. Figure 12(e) demonstrate that  $Be$  increase as the concentration difference parameter ( $\alpha_2$ ) rises. Additionally, an increase in the heat radiation parameter (Rd) and the parameter  $L$  leads to a simultaneous increase in both entropy generation and the Bejan number. This suggests that enhanced radiative heat transfer and greater diffusion increase system irreversibility, as indicated by higher entropy generation. At the same time, the increase in the Bejan number reflects that the primary cause of energy loss is heat transfer. As  $Be$  increases, The role of convective heat transfer becomes more pronounced in comparison to conductive heat transfer.

Table 2 represents frictional drag, thermal and mass transfer coefficients of tri-nanofluid by taking  $A = 0.5$ ,  $Ha = 1.5$ ,  $L = 0.1$ ,  $G = 0.5$ ,  $N = 0.5$ ,  $\delta = 0.01$ ,  $\delta_1 = 0.03$ ,  $K_E = 0.8$ ,  $T_r = 0.5$ ,  $\phi_1 = 0.35$ ,  $\phi_2 = 0.25$ ,  $\phi_3 = 0.05$  in tabular form. The results reveal that increasing the Forchheimer number or porosity parameter reduces skin friction, indicating diminished flow resistance. Simultaneously, the data show that the Nusselt number rises with enhanced radiation effects, while the Sherwood number increases with higher Schmidt numbers and chemical reaction rates, signifying improved heat and mass transfer rates.

Table 3 represents frictional drag, heat and mass transfer coefficients of di-nanofluid by taking  $A = 0.5$ ,  $Ha = 1.5$ ,  $L = 0.1$ ,  $G = 0.5$ ,  $N = 0.5$ ,  $\delta = 0.01$ ,  $\delta_1 = 0.03$ ,  $K_E = 0.8$ ,  $T_r = 0.5$ ,  $\phi_1 = 0.35$ ,  $\phi_2 = 0.25$ ,  $\phi_3 = 0$  in tabular form. The data suggest that higher values of the radiation parameter and Prandtl number enhance the Nusselt number, indicating improved heat transfer efficiency. The Sherwood number also increases with chemical reaction rates and activation energy, reflecting enhanced mass diffusion, while the skin friction values highlight the impact of boundary layer dynamics influenced by these parameters.

Table 4 shows frictional drag, heat and mass transfer coefficients of nanofluid by taking  $A = 0.5$ ,  $Ha = 1.5$ ,  $L = 0.1$ ,  $G = 0.5$ ,  $N = 0.5$ ,  $\delta = 0.01$ ,  $\delta_1 = 0.03$ ,  $K_E = 0.8$ ,  $T_r = 0.5$ ,  $\phi_1 = 0.35$ ,  $\phi_2 = 0$ ,  $\phi_3 = 0$  in tabular form. It is observed that increasing the nonlinear convection parameters leads to higher skin friction, reflecting stronger surface interactions. Concurrently, the Nusselt and Sherwood numbers display significant enhancements, illustrating improved thermal and solute transfer across the boundary layer.

The key thermophysical properties of the base fluid and nanoparticles, such as density, heat capacity, viscosity, and thermal conductivity, were utilized in the study [63,64]

Table 5 denotes the physical properties of base fluid and enhanced nanoparticles.

Table 6 denotes a detailed comparison with established findings in the literature. This quantitative comparison demonstrates the reliability of our numerical results and their alignment with established studies.

## 8 Conclusion

Thermal and material transport properties of an incompressible, viscous MHD cross nanofluid flowing through a porous medium, influenced by a vertically stretching surface, are investigated. The study considers the influence of a chemical reaction characterized by Arrhenius activation energy. Numerical computations generated results for the profiles of velocity, temperature, and concentration, as well as drag force, Nusselt number, entropy generation, and Bejan number. Additionally, validation was conducted by comparing the results with previous studies that did not account for porous media, activation energy, mixed convection, and heat sources. The significant results of this investigation are summarized as follows:

- A rise in the Brinkman number, diffusion parameter, and temperature and concentration difference parameters results in higher entropy generation. Meanwhile, the diffusion parameter, together with the parameters related to temperature and concentration differences, significantly boost the Bejan number.
- The Bejan number is influenced in opposite ways by variations in the porosity and Brinkman numbers.
- The comparative analysis between thermal efficiencies of GO, Ag, TiO<sub>2</sub> cross fluid and Ag, TiO<sub>2</sub> cross fluid is presented. It is found that the thermal efficiency of cross fluid with GO, Ag, TiO<sub>2</sub> nanoparticles are greater than that of the cross fluid with Ag, TiO<sub>2</sub> nanoparticles. It is important to note that pure cross fluid demonstrates a lower thermal conductivity than Ag, TiO<sub>2</sub> cross fluid and GO, Ag, TiO<sub>2</sub> cross fluid.
- Non-Fourier heat transfer takes place at a slower rate than Fourier transfer because of thermal memory effects influenced by thermal relaxation time. As a result, thermal variations are often diminished by the phenomenon of thermal relaxation.
- Increasing the Forchheimer number ( $F$ ) causes a decline in velocity. This occurs because a higher Forchheimer number effectively reduces the porosity of the permeable medium and enhances the drag coefficient, resulting in lower fluid velocity.

The importance and scope of this work are grounded in its potential to significantly advance thermal management technologies by leveraging the unique properties of hybrid

nanofluids. This research not only contributes to a deeper understanding of hybrid nanofluid behavior but also opens up new avenues for their application in high-performance cooling systems, energy storage solutions, and biomedical devices. The findings presented in this study align with the journal's focus on cutting-edge research in nanotechnology and its practical applications. The study's novel computational framework and results contribute to the growing body of knowledge in nanotechnology, paving the way for future advancements in both theoretical and practical applications.

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